

Optimal Stochastic Control with Recursive Cost Functionals of Stochastic Differential Systems Reflected in a Domain

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February 16, 2012

Abstract

In this paper we study the optimal stochastic control problem for stochastic differential systems reflected in a domain. The cost functional is a recursive one, which is defined via generalized backward stochastic differential equations developed by Pardoux and Zhang [17]. The value function is shown to be the viscosity solution to the associated Hamilton-Jacobi-Bellman equation, which is a fully nonlinear parabolic partial differential equation with a nonlinear Neumann boundary condition. The method of stochastic “backward semigroups” introduced by Peng [18] is adapted to our context.

AMS Subject classification: 60H99, 60H30, 35J60, 93E05, 90C39

Keywords: Hamilton-Jacobi-Bellman equation, nonlinear Neumann boundary, value function, backward stochastic differential equations, dynamic programming principle, viscosity solution

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1 Introduction

In this paper we investigate the optimal stochastic control problem for stochastic differential equations (SDEs) reflected on a given domain. The cost functional is specified by controlled generalized backward stochastic differential equations (GBSDEs) which depend on the reflections in the forward SDEs. The associated Hamilton-Jacobi-Bellman (HJB) equation turns out to have a nonlinear Neumann boundary condition. To be more detail, we aim to give the stochastic representations for the solutions of the following HJB equations with a nonlinear Neumann boundary condition:

$$\begin{cases} \frac{\partial}{\partial t} W(t, x) + H(t, x, W, DW, D^2 W) = 0, & (t, x) \in [0, T] \times D, \\ \frac{\partial}{\partial n} W(t, x) + g(t, x, W(t, x)) = 0, & 0 \leq t < T, x \in \partial D; \\ W(T, x) = \Phi(x), & x \in \bar{D}, \end{cases} \quad (1.1)$$

where at a point $x \in \partial D$, $\frac{\partial}{\partial n} = \sum_{i=1}^d \frac{\partial}{\partial x_i} \phi(x) \frac{\partial}{\partial x_i}$, and the Hamiltonian

$$H(t, x, y, p, X) = \sup_{u \in U} \left\{ \frac{1}{2} \text{tr}(\sigma \sigma^T(t, x, u) X) + p \cdot b(t, x, u) + f(t, x, y, p, \sigma, u) \right\},$$

where $t \in [0, T]$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $p \in \mathbb{R}^d$, and $X \in \mathbf{S}^d$. Here the functions b, σ, f , and Φ are supposed to satisfy (H3.1) and (H3.2), respectively. D is an open connected bounded convex subset of \mathbb{R}^d , more details in Section 3. For this, we consider the following stochastic system: For an admissible control $u(\cdot) \in \mathcal{U}$, the corresponding state process starting from $x \in \bar{D}$ at the initial time t , is governed by the following reflected SDE:

$$\begin{cases} X_s^{t,x;u} = x + \int_t^s b(r, X_r^{t,x;u}, u_r) dr + \int_t^s \sigma(r, X_r^{t,x;u}, u_r) dB_r \\ \quad + \int_t^s \nabla \phi(X_r^{t,x;u}) dK_r^{t,x;u}, \quad s \in [t, T], \\ K_s^{t,x;u} = \int_t^s I_{\{X_r^{t,x;u} \in \partial D\}} dK_r^{t,x;u}, \quad K^{t,x;u} \text{ is increasing,} \end{cases} \quad (1.2)$$

and the associated GBSDE is the following:

$$\begin{cases} -dY_s^{t,x;u} = f(s, X_s^{t,x;u}, Y_s^{t,x;u}, Z_s^{t,x;u}, u_s) ds \\ \quad + g(s, X_s^{t,x;u}, Y_s^{t,x;u}) dK_s^{t,x;u} - Z_s^{t,x;u} dB_s, \\ Y_T^{t,x;u} = \Phi(X_T^{t,x;u}). \end{cases} \quad (1.3)$$

Under standard assumptions (H3.1) and (H3.2), they have unique solutions $(X^{t,x;u}, K^{t,x;u})$ and $(Y^{t,x;u}, Z^{t,x;u})$, respectively. For any $u(\cdot) \in \mathcal{U}_{t,T}$, the cost functional is defined by

$$J(t, x; u) := Y_t^{t,x;u}, \quad (t, x) \in [0, T] \times \bar{D}. \quad (1.4)$$

The value function of our stochastic control problem is as follows:

$$W(t, x) := \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u), \quad (t, x) \in [0, T] \times \bar{D}. \quad (1.5)$$

We prove that the value function W is a viscosity solution of HJB equation (1.1).

The general formulation of backward stochastic differential equations (BSDEs) was first introduced by Pardoux and Peng [14] in 1990, who proved existence and uniqueness of adapted solutions for these equations under suitable assumptions on both the coefficients and the terminal condition. Since then they have been studied by many authors and have various applications, for example, in stochastic control, finance and partial differential equations (PDEs). The reader is referred to, among others, EL Karoui, Peng and Quenez [9], Darling and Pardoux [6], Pardoux and Peng [15], Peng [18], [19], Hu [10], and Delbaen and Tang [8]. Stochastic differential equations with reflecting boundary conditions have been studied by many authors, see Lions [11], Lions and Sznitman [12], Menaldi [13], Pardoux and Williams [16], Saisho [20], among others. Note that in the work of Pardoux and Zhang [17], a new class of BSDEs, which is driven by, in addition to a Brownian motion, a continuous increasing process, is studied and used to give a probabilistic formula for the solution of a system of parabolic or elliptic semi-linear partial differential equation with a nonlinear Neumann boundary condition. There are many related works, and the reader is referred to Boufoussia and Van Casterenb [2], Day [7], etc.

The BSDE method developed by Peng [18] and [19] for the dynamic programming of optimal stochastic control, is extended into our context. We shall also adapt the arguments of Buckdahn and Li [4] to show that our value function W (see (3.7)), which is introduced as the essential supremum over a class of random variables, is deterministic (see Proposition 3.1). Such an effort is not trivial since the cost is also driven by the increasing process to incorporate the reflection of the system state on the boundary of the given domain. It is crucial for us to show that it is the continuous viscosity solution to the associated HJB equation subject to a nonlinear Neumann boundary condition (Theorem 4.1). It allows us to prove the dynamic programming principle (DPP in short, see Theorem 3.1) in a straight forward way by adapting to GBSDEs the method of stochastic backward semigroups introduced by Peng [18]. However, the proof becomes more technical due to the Neumann boundary condition. We have to improve the estimate on GBSDE of Pardoux and Zhang [17]. We could prove that, in the Markovian framework and under standard assumptions, the dependence of the solution on the random initial value of the driving SDE is locally Lipschitz (Proposition 5.2), and the increasing process has a new important estimate (Proposition 5.3). Furthermore, our proof of Theorem 4.1, which states that the value function W is a continuous viscosity solution of the associated HJB equations with the Neumann boundary condition, differs heavily from those counterpart of either Buckdahn and Li [4] or Peng [18]. For the detailed complication, the reader is referred to among others Lemmas 4.2 and 4.3 and the constructions of BSDEs (4.10), (4.12), (4.23) and (4.24). In our context, the coefficients of the problem are not necessarily continuous in the control variable u , and the control u may take values in a possibly noncompact space U .

Our paper is organized as follows. In Section 2, we recall some elements of the theory of BSDEs and GBSDEs. In Section 3, we introduce the setting of our stochastic

control problem, the nonlinear Neumann boundary condition, and the value function W . We prove that W is deterministic and satisfies the DPP, which implies that W is continuous. In Section 4, we prove that W is a viscosity solution of the associated HJB equation with a nonlinear Neumann boundary condition. For the sake of readers, some necessary basic properties of GBSDEs associated with forward reflected SDEs, are documented in the appendix (Section 5), where Propositions 5.2 and 5.3 contain new results on GBSDEs.

2 Preliminaries

Consider the Wiener space (Ω, \mathcal{F}, P) , where Ω is the set of continuous functions from $[0, T]$ to \mathbb{R}^d starting from 0 ($\Omega = C_0([0, T]; \mathbb{R}^d)$), \mathcal{F} the completed Borel σ -algebra over Ω , and P the Wiener measure. Let B be the canonical process: $B_s(\omega) = \omega_s$, $s \in [0, T]$, $\omega \in \Omega$. By $\mathbb{F} = \{\mathcal{F}_s, 0 \leq s \leq T\}$ we denote the natural filtration generated by $\{B_s\}_{0 \leq s \leq T}$ and augmented by all P -null sets, i.e.,

$$\mathcal{F}_s = \sigma\{B_r, r \leq s\} \vee \mathcal{N}, \quad s \in [0, T],$$

where \mathcal{N} is the set of all P -null subsets, and $T > 0$ a fixed real time horizon. For any $n \geq 1$, $|z|$ denotes the Euclidean norm of $z \in \mathbb{R}^n$. Introduce the following two spaces of processes: $\mathcal{S}^2(0, T; \mathbb{R})$ is the collection of $(\psi_t)_{0 \leq t \leq T}$ which is a real-valued adapted càdlàg process such that $E[\sup_{0 \leq t \leq T} |\psi_t|^2] < +\infty$, and $\mathcal{H}^2(0, T; \mathbb{R}^n)$ is the collection of $(\psi_t)_{0 \leq t \leq T}$ which is an \mathbb{R}^n -valued progressively measurable process such that $\|\psi\|_2^2 = E[\int_0^T |\psi_t|^2 dt] < +\infty$.

We make the following assumptions on g throughout the paper.

(A1) The function $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ has the property that $(g(t, y, z))_{t \in [0, T]}$ is progressively measurable for each (y, z) in $\mathbb{R} \times \mathbb{R}^d$. There exists a constant $C \geq 0$ such that, P -a.s., for all $t \in [0, T]$, $y_1, y_2 \in \mathbb{R}$, $z_1, z_2 \in \mathbb{R}^d$,

$$|g(t, y_1, z_1) - g(t, y_2, z_2)| \leq C(|y_1 - y_2| + |z_1 - z_2|).$$

(A2) $g(\cdot, 0, 0) \in \mathcal{H}^2(0, T; \mathbb{R})$.

The following result is due to Pardoux and Peng [14] on BSDEs.

Lemma 2.1. *Under assumptions (A1) and (A2), the BSDE*

$$y_t = \xi + \int_t^T g(s, y_s, z_s) ds - \int_t^T z_s dB_s, \quad 0 \leq t \leq T, \quad (2.1)$$

has a unique adapted solution $(y, z) \in \mathcal{S}^2(0, T; \mathbb{R}) \times \mathcal{H}^2(0, T; \mathbb{R}^d)$ for any random variable $\xi \in L^2(\mathcal{O}, \mathcal{F}_T, P)$.

We need the following comparison theorem on BSDEs, which can be found in El Karoui, Peng, and Quenez [9].

Lemma 2.2. *(Comparison Theorem) Let g_1 and g_2 satisfy (A1) and (A2) and $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, P)$. Denote the solution of BSDE (2.1) by (y^1, z^1) for the data $(\xi, g) = (\xi_1, g_1)$ and by (y^2, z^2) for the data $(\xi, g) = (\xi_2, g_2)$. Then we have*

- (i) (*Monotonicity*) If $\xi_1 \geq \xi_2$ and $g_1 \geq g_2$, a.s., then $y_t^1 \geq y_t^2$, a.s. for all $t \in [0, T]$.
(ii) (*Strict Monotonicity*) If $P(\xi_1 > \xi_2) > 0$ in addition to (i), then $P\{y_t^1 > y_t^2\} > 0$ for $t \in [0, T]$, and in particular, $y_0^1 > y_0^2$.

Lemma 2.3. Suppose that the random field $g : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ satisfies (A1) and (A2), and the two random variables $\xi_1, \xi_2 \in L^2(\Omega, \mathcal{F}_T, P)$. Define

$$g_i(s, y, z) = g(s, y, z) + \varphi_i(s), \quad (s, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d,$$

where $\varphi_i \in \mathcal{H}^2(0, T; \mathbb{R})$ for $i = 1, 2$. Let (y^1, z^1) be the solution to BSDE (2.1) for the data $(\xi, g) = (\xi_1, g_1)$ and (y^2, z^2) for the data $(\xi, g) = (\xi_2, g_2)$. Then we have the following estimate

$$\begin{aligned} & |y_t^1 - y_t^2|^2 + \frac{1}{2} E \left[\int_t^T e^{\beta(s-t)} (|y_s^1 - y_s^2|^2 + |z_s^1 - z_s^2|^2) ds \mid \mathcal{F}_t \right] \\ & \leq E \left[e^{\beta(T-t)} |\xi_1 - \xi_2|^2 \mid \mathcal{F}_t \right] + E \left[\int_t^T e^{\beta(s-t)} |\varphi_1(s) - \varphi_2(s)|^2 ds \mid \mathcal{F}_t \right], \quad \forall t \in [0, T], \end{aligned}$$

where $\beta = 16(1 + C^2)$.

For the proof the reader is referred to El Karoui, Peng, and Quenez [9] and Peng [18].

Let $\{A_t, t \geq 0\}$ be a continuous increasing \mathbb{F} -progressively measurable scalar process, satisfying $A_0 = 0$ and $E[e^{\mu A_T}] < \infty$ for all $\mu > 0$. We are given a final condition $\xi \in L^2(\Omega, \mathcal{F}_T, P)$ such that $E(e^{\mu A_T} |\xi|^2) < \infty$ for all $\mu > 0$, and two random fields $f : \Omega \times [0, T] \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ and $g : \Omega \times [0, T] \times \mathbb{R} \rightarrow \mathbb{R}$ satisfying,

(H2.1) (i) The processes $f(\cdot, y, z)$ and $g(\cdot, y)$ are progressively measurable and

$$E \left[\int_0^T e^{\mu A_t} |f(t, 0, 0)|^2 dt \right] + E \left[\int_0^T e^{\mu A_t} |g(t, 0)|^2 dA_t \right] < \infty \text{ for all } \mu > 0;$$

(ii) there is a constant C such that for all $(t, y, z) \in [0, T] \times \mathbb{R} \times \mathbb{R}^d$,

$$|f(t, y, z) - f(t, y', z')| \leq C(|y - y'| + |z - z'|);$$

(iii) there exists a constant C such that for all $(t, y) \in [0, T] \times \mathbb{R}$,

$$|g(t, y) - g(t, y')| \leq C|y - y'|.$$

A solution to the following GBSDE

$$Y_t = \xi + \int_t^T f(s, Y_s, Z_s) ds + \int_t^T g(s, Y_s) dA_s - \int_t^T Z_s dB_s, \quad 0 \leq t \leq T, \quad (2.2)$$

is a pair $\{(Y_t, Z_t), 0 \leq t \leq T\}$ of progressively measurable processes taking values in $\mathbb{R} \times \mathbb{R}^d$ which satisfies equation (2.2) and

$$E \left[\sup_{0 \leq t \leq T} |Y_t|^2 \right] + E \left[\int_0^T |Z_t|^2 dt \right] < \infty, \quad 0 \leq t \leq T. \quad (2.3)$$

From Theorem 1.6 and Proposition 1.1 of Pardoux and Zhang [17], we have the following two lemmas.

Lemma 2.4. *Let (H2.1) be satisfied. Then GBSDE (2.2) has a unique solution (Y, Z) .*

Lemma 2.5. *Under the assumption (H2.1), we have for any $\mu > 0$*

$$\begin{aligned} & E \left[\sup_{0 \leq t \leq T} e^{\mu A_t} |Y_t|^2 + \int_0^T e^{\mu A_t} |Y_t|^2 dA_t + \int_0^T e^{\mu A_t} |Z_t|^2 dt \right] \\ & \leq CE \left[e^{\mu A_T} |\xi|^2 + \int_0^T e^{\mu A_t} |f(t, 0, 0)|^2 dt + \int_0^T e^{\mu A_t} |g(t, 0)|^2 dA_t \right] \end{aligned} \quad (2.4)$$

for a positive constant C , which depends on the Lipschitz constant of f and g , μ , and T .

Let two sets of data (ξ, f, g, A) and (ξ', f', g', A') satisfy assumption (H2.1). Let (Y, Z) is a solution to GBSDE (2.2) for data (ξ, f, g, A) and (Y', Z') for data (ξ', f', g', A') . Define

$$(\bar{Y}, \bar{Z}, \bar{\xi}, \bar{f}, \bar{g}, \bar{A}) = (Y - Y', Z - Z', \xi - \xi', f - f', g - g', A - A').$$

The following two lemmas are borrowed from Proposition 1.2 and Theorem 1.4 of Pardoux and Zhang [17], respectively.

Lemma 2.6. *For any $\mu > 0$, there exists a constant C such that*

$$\begin{aligned} & E \left[\sup_{0 \leq t \leq T} e^{\mu k_t} |\bar{Y}_t|^2 + \int_0^T e^{\mu k_t} |\bar{Z}_t|^2 dt \right] \\ & \leq CE \left[e^{\mu k_T} |\bar{\xi}|^2 + \int_0^T e^{\mu k_t} |\bar{f}(t, Y_t, Z_t)|^2 dt \right. \\ & \quad \left. + \int_0^T e^{\mu k_t} |\bar{g}(t, Y_t)|^2 dA'_t + \int_0^T e^{\mu k_t} |g(t, Y_t)|^2 d\|\bar{A}\|_t \right], \end{aligned} \quad (2.5)$$

where $k_t := \|\bar{A}\|_t + A'_t$, and $\|\bar{A}\|_t$ is the total variation of the process \bar{A} on the interval $[0, t]$.

For the particular case $A \equiv A'$, we have

Lemma 2.7. *(Comparison Theorem) Assume that $\xi \leq \xi'$, $f(t, y, z) \leq f'(t, y, z)$, and $g(t, y) \leq g'(t, y)$, for all $(y, z) \in \mathbb{R} \times \mathbb{R}^d$, $dP \times dt$, a.s. Then $Y_t \leq Y'_t$, $0 \leq t \leq T$, a.s. Moreover, if $Y_0 = Y'_0$, then $Y_t = Y'_t$, $0 \leq t \leq T$, a.s. In particular, if in addition either $P(\xi < \xi') > 0$ or $f(t, y, z) < f'(t, y, z)$ for any $(y, z) \in \mathbb{R} \times \mathbb{R}^d$ holds on a set of positive $dt \times dP$ measure, or $g(t, y) < g'(t, y)$ for any $y \in \mathbb{R}$ holds on a set of positive $dA_t \times dP$ measure, then $Y_0 < Y'_0$.*

3 Formulation of the problem and related DPP

Let D be an open connected bounded convex subset of \mathbb{R}^d , which is such that for a function $\phi \in C_b^2(\mathbb{R}^d)$, $D = \{\phi > 0\}$, $\partial D = \{\phi = 0\}$, and $|\nabla \phi(x)| = 1, x \in \partial D$. Note that at any boundary point $x \in \partial D$, $\nabla \phi(x)$ is a unit normal vector to the boundary, pointing towards the interior of D . Let U be a metric space. An admissible control

process is a U -valued \mathbb{F} -progressively measurable process. The set of all admissible control processes is denoted by \mathcal{U} .

For an admissible control $u(\cdot) \in \mathcal{U}$, the corresponding state process starting from $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \bar{D})$ at the initial time t , is governed by the following reflected SDE:

$$\begin{cases} X_s^{t,\zeta;u} &= \zeta + \int_t^s b(r, X_r^{t,\zeta;u}, u_r)dr + \int_t^s \sigma(r, X_r^{t,\zeta;u}, u_r)dB_r \\ &+ \int_t^s \nabla \phi(X_r^{t,\zeta;u})dK_r^{t,\zeta;u}, \quad s \in [t, T], \\ K_s^{t,\zeta;u} &= \int_t^s I_{\{X_r^{t,\zeta;u} \in \partial D\}} dK_r^{t,\zeta;u}, \quad K^{t,\zeta;u} \text{ is increasing.} \end{cases} \quad (3.1)$$

Here, we have made the following assumption on the drift $b : [0, T] \times \mathbb{R}^d \times U \rightarrow \mathbb{R}^d$ and the diffusion $\sigma : [0, T] \times \mathbb{R}^d \times U \rightarrow \mathbb{R}^{d \times d}$:

- (H3.1) (i) For every fixed $x \in \mathbb{R}^n$, $u \in U$, $b(\cdot, x, u)$ and $\sigma(\cdot, x, u)$ are continuous in t ;
(ii) there exists a $C > 0$ such that, for all $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $u \in U$,
 $|b(t, x, u) - b(t, x', u)| + |\sigma(t, x, u) - \sigma(t, x', u)| \leq C|x - x'|$;
(iii) there is some $C > 0$ such that, for all $t \in [0, T]$, $u \in U$ and $x \in \mathbb{R}^n$,
 $|b(t, x, u)| + |\sigma(t, x, u)| \leq C(1 + |x|)$.

Therefore, in view of Proposition 5.1 in the appendix, SDE (3.1) has a unique strong solution $(X^{t,\zeta;u}, K^{t,\zeta;u})$ for any $u(\cdot) \in \mathcal{U}$. Moreover, for any $t \in [0, T]$, $u(\cdot) \in \mathcal{U}$, and $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \bar{D})$, we have

$$\begin{aligned} E \left[\sup_{s \in [t, T]} |X_s^{t,\zeta;u} - X_s^{t,\zeta';u}|^4 | \mathcal{F}_t \right] &\leq C|\zeta - \zeta'|^4, \\ E \left[\sup_{s \in [t, T]} |X_s^{t,\zeta;u}|^4 | \mathcal{F}_t \right] &\leq C(1 + |\zeta|^4). \end{aligned} \quad (3.2)$$

Here, the constant C depends only on the Lipschitz and the linear growth constants of b and σ with respect to x .

Assume that three functions

$$\Phi : \mathbb{R}^d \rightarrow \mathbb{R}, \quad f : [0, T] \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times U \rightarrow \mathbb{R}, \quad g : [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$$

satisfy the following conditions:

- (H3.2) (i) For every fixed $(x, y, z, u) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \times U$, $f(\cdot, x, y, z, u)$ is continuous in t ; $g(\cdot) \in C^{1,2,2}([0, T] \times \mathbb{R}^d \times \mathbb{R})$; and there exists a constant $C > 0$ such that, for all $t \in [0, T]$, $x, x' \in \mathbb{R}^n$, $y, y' \in \mathbb{R}$, $z, z' \in \mathbb{R}^d$, $u \in U$,
 $|f(t, x, y, z, u) - f(t, x', y', z', u)| + |g(t, x, y) - g(t, x', y')|$
 $\leq C(|x - x'| + |y - y'| + |z - z'|)$;
(ii) there is a constant $C > 0$ such that, for all $x, x' \in \mathbb{R}^n$,
 $|\Phi(x) - \Phi(x')| \leq C|x - x'|$.
(iii) there exists some $C > 0$ such that, for all $0 \leq t \leq T$, $u \in U$, and $x \in \mathbb{R}^n$,
 $|f(t, x, 0, 0, u)| \leq C(1 + |x|)$.

Then, obviously, g and Φ also have the global linear growth condition in x : there exists some $C > 0$ such that, for all $0 \leq t \leq T$, and $x \in \mathbb{R}^n$,

$$|g(t, x, 0)| + |\Phi(x)| \leq C(1 + |x|).$$

For any $u(\cdot) \in \mathcal{U}$, and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \bar{D})$, the mappings $\xi := \Phi(X_T^{t, \zeta; u})$, $\tilde{g}(s, y) := g(s, X_s^{t, \zeta; u}, y)$ and $\tilde{f}(s, y, z) := f(s, X_s^{t, \zeta; u}, y, z, u_s)$ satisfy the conditions $H(2.1)$ on the interval $[t, T]$. Therefore, there is a unique solution to the following GBSDE:

$$\begin{cases} -dY_s^{t, \zeta; u} &= f(s, X_s^{t, \zeta; u}, Y_s^{t, \zeta; u}, Z_s^{t, \zeta; u}, u_s) ds \\ &\quad + g(s, X_s^{t, \zeta; u}, Y_s^{t, \zeta; u}) dK_s^{t, \zeta; u} - Z_s^{t, \zeta; u} dB_s, \\ Y_T^{t, \zeta; u} &= \Phi(X_T^{t, \zeta; u}), \end{cases} \quad (3.3)$$

where $(X^{t, \zeta; u}, K^{t, \zeta; u})$ solves the reflected SDE (3.1).

Moreover, similar to Proposition 5.2, there exists some constant $C > 0$ such that, for all $0 \leq t \leq T$, $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \bar{D})$, $u(\cdot) \in \mathcal{U}$, P -a.s.,

$$\begin{aligned} \text{(i)} \quad & |Y_t^{t, \zeta; u} - Y_t^{t, \zeta'; u}| \leq C[|\zeta - \zeta'| + |\zeta - \zeta'|^{\frac{1}{2}}]; \\ \text{(ii)} \quad & |Y_t^{t, \zeta; u}| \leq C(1 + |\zeta|). \end{aligned} \quad (3.4)$$

We now introduce the following sets of admissible controls.

Definition 3.1. An admissible control process $u = \{u_r, r \in [t, s]\}$ on $[t, s]$ (with $s \in (t, T]$) is an \mathcal{F}_r -progressively measurable process taking values in U . The set of all admissible controls on $[t, s]$ is denoted by $\mathcal{U}_{t,s}$. We identify two processes u and \bar{u} in $\mathcal{U}_{t,s}$ and write $u \equiv \bar{u}$ on $[t, s]$, if $P\{u = \bar{u} \text{ a.e. in } [t, s]\} = 1$.

For any $u(\cdot) \in \mathcal{U}_{t,T}$, the value of the associated cost functional is given by

$$J(t, x; u) := Y_t^{t, x; u}, \quad (t, x) \in [0, T] \times \bar{D}, \quad (3.5)$$

where the process $Y^{t, x; u}$ is defined by GBSDE (3.3).

From Theorem 5.1, we have

$$J(t, \zeta; u) = Y_t^{t, \zeta; u}, \quad (t, \zeta) \in [0, T] \times L^2(\Omega, \mathcal{F}_t, P; \bar{D}). \quad (3.6)$$

We define the value function of our stochastic control problem as follows:

$$W(t, x) := \operatorname{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u), \quad (t, x) \in [0, T] \times \bar{D}. \quad (3.7)$$

Under assumptions (H3.1) and (H3.2), the value function W is well-defined on $[0, T] \times D$, and its values at time t are bounded and \mathcal{F}_t -measurable random variables. In fact, they are all deterministic. We have

Proposition 3.1. For any $(t, x) \in [0, T] \times \bar{D}$, we have $W(t, x) = E[W(t, x)]$, P -a.s. Let $W(t, x)$ equal to its deterministic version $E[W(t, x)]$. Then $W : [0, T] \times \bar{D} \rightarrow \mathbb{R}$ is a deterministic function.

Proof. The proof is an adaptation of relevant arguments of Buckdahn and Li [4]. Let H be the Cameron–Martin space of all absolutely continuous elements $h \in \Omega$ whose derivative \dot{h} is in $L^2([0, T], \mathbb{R}^d)$.

For any $h \in H$, we define $\tau_h \omega := \omega + h$, $\omega \in \Omega$. Obviously, $\tau_h : \Omega \rightarrow \Omega$ is a bijection with the inverse τ_h^{-1} . The law is given by

$$P \circ [\tau_h^{-1}] = \exp\left\{\int_0^T \dot{h}_s dB_s - \frac{1}{2} \int_0^T |\dot{h}_s|^2 ds\right\} P.$$

Fix any $(t, x) \in [0, T] \times \bar{D}$, and define $H_t := \{h \in H | h(\cdot) = h(\cdot \wedge t)\}$. The rest of the proof is divided into the following three steps:

Step 1. For any $u \in \mathcal{U}_{t,T}$ and $h \in H_t$, $J(t, x; u)(\tau_h) = J(t, x; u(\tau_h))$, P -a.s.

Indeed, the τ_h -shifted reflected SDE (3.1) (with $\zeta = x$) is the same reflected SDE (3.1) with u being substituted into the τ_h -shifted control process $u(\tau_h)$. From the uniqueness of the solution of the reflected SDE (3.1), we get $X_s^{t,x;u}(\tau_h) = X_s^{t,x;u(\tau_h)}$ and $K_s^{t,x;u}(\tau_h) = K_s^{t,x;u(\tau_h)}$ for $s \in [t, T]$ P -a.s. Furthermore, by a similar shift argument and the associated Girsanov transformation, we get from the uniqueness of the solution of GBSDE (3.3) that

$$\begin{aligned} Y_s^{t,x;u}(\tau_h) &= Y_s^{t,x;u(\tau_h)} \text{ for any } s \in [t, T], \text{ } P\text{-a.s.}, \\ Z_s^{t,x;u}(\tau_h) &= Z_s^{t,x;u(\tau_h)}, \text{ dsd } P\text{-a.e. on } [t, T] \times \Omega. \end{aligned}$$

It means

$$J(t, x; u)(\tau_h) = J(t, x; u(\tau_h)), \text{ } P\text{-a.s.}$$

Step 2. For all $h \in H_t$ we have

$$\{\text{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u)\}(\tau_h) = \text{esssup}_{u \in \mathcal{U}_{t,T}} \{J(t, x; u)(\tau_h)\}, \text{ } P\text{-a.s.}$$

Indeed, define

$$W(t, x) := \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u).$$

we have $W(t, x) \geq J(t, x; u)$, and thus $W(t, x)(\tau_h) \geq J(t, x; u)(\tau_h)$, P -a.s., for all $u \in \mathcal{U}_{t,T}$. On the other hand, for any random variable ζ satisfying $\zeta \geq J(t, x; u)(\tau_h)$, and hence also $\zeta(\tau_{-h}) \geq J(t, x; u)$, P -a.s., for all $u \in \mathcal{U}_{t,T}$, we have $\zeta(\tau_{-h}) \geq W(t, x)$, P -a.s., i.e., $\zeta \geq W(t, x)(\tau_h)$, P -a.s. Consequently,

$$W(t, x)(\tau_h) = \text{esssup}_{u \in \mathcal{U}_{t,T}} \{J(t, x; u)(\tau_h)\}, \text{ } P\text{-a.s.}$$

Step 3. $W(t, x)$ is invariant with respect to the shift τ_h , i.e.,

$$W(t, x)(\tau_h) = W(t, x), \text{ } P\text{-a.s.}, \text{ for any } h \in H.$$

Indeed, from Step 1 to Step 2, we have, for any $h \in H_t$,

$$\begin{aligned} W(t, x)(\tau_h) &= \text{esssup}_{u \in \mathcal{U}_{t,T}} \{J(t, x; u)(\tau_h)\} \\ &= \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u(\tau_h)) \\ &= \text{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u) \\ &= W(t, x), \text{ } P\text{-a.s.}, \end{aligned}$$

where we have used $\{u(\tau_h)|u(\cdot) \in \mathcal{U}_{t,T}\} = \mathcal{U}_{t,T}$ so as to obtain the 3rd equality. Therefore, $W(t, x)(\tau_h) = W(t, x)$, P -a.s. for any $h \in H_t$. Since $W(t, x)$ is \mathcal{F}_t -measurable, it holds for all $h \in H$. Indeed, since $\Omega = C_0([0, T]; \mathbb{R}^d)$, by the definition of the filtration, the \mathcal{F}_t -measurable random variable $W(t, x)(\omega)$, $\omega \in \Omega$, only depends on the restriction of ω to the time interval $[0, t]$.

The result of Step 3, combined with the following Lemma 3.1 completes the proof. \square

The following is borrowed from Buckdahn and Li [4, Lemma 3.4].

Lemma 3.1. *Let ζ be a random variable defined on the Wiener space $(\Omega, \mathcal{F}_T, P)$ such that $\zeta(\tau_h) = \zeta$ P -a.s. for any $h \in H$. Then $\zeta = E\zeta$ P -a.s.*

As an immediate result of (3.4) and (3.7), the value function W has the following property .

Lemma 3.2. *There exists a constant $C > 0$ such that, for all $(t, x, x') \in [0, T] \times \bar{D} \times \bar{D}$,*

$$\begin{aligned} \text{(i)} \quad & |W(t, x) - W(t, x')| \leq C[|x - x'| + |x - x'|^{\frac{1}{2}}]; \\ \text{(ii)} \quad & |W(t, x)| \leq C(1 + |x|). \end{aligned} \tag{3.8}$$

We now study the (generalized) DPP for our stochastic control problem (3.1), (3.3), and (3.7). For this we have to define the family of (backward) semigroups related with GBSDE (3.3). Peng [18] first introduced the notion of backward stochastic semigroups to study the DPP for the optimal stochastic control of SDEs. In what follows, it is extended to the optimal stochastic control of *reflected* SDEs.

Given the initial data (t, x) , a positive number $\delta \leq T - t$, an admissible control $u(\cdot) \in \mathcal{U}_{t, t+\delta}$, and a random variable $\eta \in L^2(\Omega, \mathcal{F}_{t+\delta}, P; \mathbb{R})$, we define

$$G_{s, t+\delta}^{t, x; u}[\eta] := \tilde{Y}_s^{t, x; u}, \quad s \in [t, t + \delta], \tag{3.9}$$

where $(\tilde{Y}_s^{t, x; u}, \tilde{Z}_s^{t, x; u})_{t \leq s \leq t+\delta}$ is the solution of the following GBSDE on the time interval $[t, t + \delta]$:

$$\begin{cases} -d\tilde{Y}_s^{t, x; u} &= f(s, X_s^{t, x; u}, \tilde{Y}_s^{t, x; u}, \tilde{Z}_s^{t, x; u}, u_s) ds + g(s, X_s^{t, x; u}, \tilde{Y}_s^{t, x; u}) dK_s^{t, x; u} \\ &\quad - \tilde{Z}_s^{t, x; u} dB_s, \quad s \in [t, t + \delta]; \\ \tilde{Y}_{t+\delta}^{t, x; u} &= \eta, \end{cases}$$

and $(X^{t, x; u}, K^{t, x; u})$ is the solution of reflected SDE (3.1). Then, obviously, for the solution $(Y^{t, x; u}, Z^{t, x; u})$ of GBSDE (3.3), we have

$$G_{t, T}^{t, x; u}[\Phi(X_T^{t, x; u})] = G_{t, t+\delta}^{t, x; u}[Y_{t+\delta}^{t, x; u}]. \tag{3.10}$$

Furthermore,

$$\begin{aligned} J(t, x; u) &= Y_t^{t, x; u} = G_{t, T}^{t, x; u}[\Phi(X_T^{t, x; u})] = G_{t, t+\delta}^{t, x; u}[Y_{t+\delta}^{t, x; u}] \\ &= G_{t, t+\delta}^{t, x; u}[J(t + \delta, X_{t+\delta}^{t, x; u}; u)]. \end{aligned}$$

Remark 3.1. *If both f and g do not depend on (y, z) , we have*

$$G_{s, t+\delta}^{t, x; u}[\eta] = E \left[\eta + \int_s^{t+\delta} f(r, X_r^{t, x; u}, u_r) dr + \int_s^{t+\delta} g(r, X_r^{t, x; u}) dK_r^{t, x; u} \middle| \mathcal{F}_s \right], \quad s \in [t, t+\delta].$$

Theorem 3.1. *Under assumptions (H3.1) and (H3.2), the value function W satisfies the following DPP: for any $0 \leq t < t + \delta \leq T$, $x \in \bar{D}$,*

$$W(t, x) = \operatorname{esssup}_{u \in \mathcal{U}_{t, t+\delta}} G_{t, t+\delta}^{t, x; u} [W(t + \delta, X_{t+\delta}^{t, x; u})]. \quad (3.11)$$

To simplify our exposition, define

$$I_\delta(t, x, u) := G_{t, t+\delta}^{t, x; u} [W(t + \delta, X_{t+\delta}^{t, x; u})]$$

and

$$W_\delta(t, x) := \operatorname{esssup}_{u \in \mathcal{U}_{t, t+\delta}} I_\delta(t, x, u) = \operatorname{esssup}_{u \in \mathcal{U}_{t, t+\delta}} G_{t, t+\delta}^{t, x; u} [W(t + \delta, X_{t+\delta}^{t, x; u})].$$

Similar to the proof of Proposition 3.1, we have

Lemma 3.3. *$W_\delta(t, x)$ is deterministic for any $0 \leq t < t + \delta \leq T$, $x \in \bar{D}$.*

The proof of Theorem 3.1 is reduced to the following two lemmas.

Lemma 3.4. *$W_\delta(t, x) \leq W(t, x)$, $0 \leq t < t + \delta \leq T$, $x \in \bar{D}$.*

Proof. For $u_1(\cdot) \in \mathcal{U}_{t, t+\delta}$ and $u_2(\cdot) \in \mathcal{U}_{t+\delta, T}$, we define

$$u_1 \oplus u_2 := u_1 \mathbf{1}_{[t, t+\delta]} + u_2 \mathbf{1}_{(t+\delta, T]},$$

which lies in $\mathcal{U}_{t, T}$. Note that there exists a sequence $\{u_i^1, i \geq 1\} \subset \mathcal{U}_{t, t+\delta}$ such that

$$W_\delta(t, x) = \operatorname{esssup}_{u_1 \in \mathcal{U}_{t, t+\delta}} I_\delta(t, x, u_1) = \sup_{i \geq 1} I_\delta(t, x, u_i^1), \quad P\text{-a.s.}$$

For any $\varepsilon > 0$, we define

$$\tilde{\Gamma}_i := \{W_\delta(t, x) \leq I_\delta(t, x, u_i^1) + \varepsilon\} \in \mathcal{F}_t, \quad i \geq 1.$$

Then the following mutually disjoint events

$$\Gamma_1 := \tilde{\Gamma}_1, \quad \Gamma_i := \tilde{\Gamma}_i \setminus (\cup_{l=1}^{i-1} \tilde{\Gamma}_l) \in \mathcal{F}_t, \quad i \geq 2$$

form a (Ω, \mathcal{F}_t) -partition. It is obvious that

$$u_1^\varepsilon := \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} u_i^1 \in \mathcal{U}_{t, t+\delta}.$$

Moreover, from the uniqueness of the solution of the forward-backward SDE (FBSDE), we have

$$I_\delta(t, x, u_1^\varepsilon) = \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} I_\delta(t, x, u_i^1), \quad P\text{-a.s.}$$

Hence,

$$\begin{aligned} W_\delta(t, x) &\leq \sum_{i \geq 1} \mathbf{1}_{\Gamma_i} I_\delta(t, x, u_i^1) + \varepsilon = I_\delta(t, x, u_1^\varepsilon) + \varepsilon \\ &= G_{t, t+\delta}^{t, x; u_1^\varepsilon} [W(t + \delta, X_{t+\delta}^{t, x; u_1^\varepsilon})] + \varepsilon, \quad P\text{-a.s.} \end{aligned} \quad (3.12)$$

On the other hand, from the definition of $W(t + \delta, y)$ we have, for any $y \in \bar{D}$,

$$W(t + \delta, y) = \operatorname{esssup}_{u_2 \in \mathcal{U}_{t+\delta, T}} J(t + \delta, y; u_2), \quad P\text{-a.s.}$$

Finally, since there exists a constant $C \in \mathbb{R}$ such that for any $y, y' \in \bar{D}$,

$$\begin{aligned} \text{(i)} \quad & |W(t + \delta, y) - W(t + \delta, y')| \leq C \left(|y - y'| + |y - y'|^{\frac{1}{2}} \right); \\ \text{(ii)} \quad & |J(t + \delta, y, u_2) - J(t + \delta, y', u_2)| \leq C \left(|y - y'| + |y - y'|^{\frac{1}{2}} \right) \quad P\text{-a.s.}, \end{aligned} \quad (3.13)$$

for any $u_2 \in \mathcal{U}_{t+\delta, T}$,

(see Lemma 3.2(i) and (3.4)(i)) we can prove by approximating $X_{t+\delta}^{t, x; u_1^\varepsilon}$ that

$$W(t + \delta, X_{t+\delta}^{t, x; u_1^\varepsilon}) \leq \operatorname{esssup}_{u_2 \in \mathcal{U}_{t+\delta, T}} J(t + \delta, X_{t+\delta}^{t, x; u_1^\varepsilon}; u_2), \quad P\text{-a.s.}$$

To estimate the right side of the above inequality we notice that there exists some sequence $\{u_j^2, j \geq 1\} \subset \mathcal{U}_{t+\delta, T}$ such that

$$\operatorname{esssup}_{u_2 \in \mathcal{U}_{t+\delta, T}} J(t + \delta, X_{t+\delta}^{t, x; u_1^\varepsilon}; u_2) = \sup_{j \geq 1} J(t + \delta, X_{t+\delta}^{t, x; u_1^\varepsilon}; u_j^2), \quad P\text{-a.s.}$$

Then, putting $\tilde{\Delta}_j := \{\operatorname{esssup}_{u_2 \in \mathcal{U}_{t+\delta, T}} J(t + \delta, X_{t+\delta}^{t, x; u_1^\varepsilon}; u_2) \leq J(t + \delta, X_{t+\delta}^{t, x; u_1^\varepsilon}; u_j^2) + \varepsilon\} \in \mathcal{F}_{t+\delta}$, $j \geq 1$; we have with $\Delta_1 := \tilde{\Delta}_1$, $\Delta_j := \tilde{\Delta}_j \setminus (\cup_{l=1}^{j-1} \tilde{\Delta}_l) \in \mathcal{F}_{t+\delta}$, $j \geq 2$, an $(\Omega, \mathcal{F}_{t+\delta})$ -partition and $u_2^\varepsilon := \sum_{j \geq 1} \mathbf{1}_{\Delta_j} u_j^2 \in \mathcal{U}_{t+\delta, T}$. Therefore, from the uniqueness of the solution of our reflected SDE and GBSDE, we have

$$\begin{aligned} J(t + \delta, X_{t+\delta}^{t, x; u_1^\varepsilon}; u_2^\varepsilon) &= Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t, x; u_1^\varepsilon}; u_2^\varepsilon} \quad (\text{see (3.6)}) \\ &= \sum_{j \geq 1} \mathbf{1}_{\Delta_j} Y_{t+\delta}^{t+\delta, X_{t+\delta}^{t, x; u_1^\varepsilon}; u_j^2} \\ &= \sum_{j \geq 1} \mathbf{1}_{\Delta_j} J(t + \delta, X_{t+\delta}^{t, x; u_1^\varepsilon}; u_j^2), \quad P\text{-a.s.} \end{aligned}$$

Thus,

$$\begin{aligned} W(t + \delta, X_{t+\delta}^{t, x; u_1^\varepsilon}) &\leq \operatorname{esssup}_{u_2 \in \mathcal{U}_{t+\delta, T}} J(t + \delta, X_{t+\delta}^{t, x; u_1^\varepsilon}; u_2) \\ &\leq \sum_{j \geq 1} \mathbf{1}_{\Delta_j} Y_{t+\delta}^{t, x; u_1^\varepsilon \oplus u_j^2} + \varepsilon \\ &= Y_{t+\delta}^{t, x; u_1^\varepsilon \oplus u_2^\varepsilon} + \varepsilon \\ &= Y_{t+\delta}^{t, x; u^\varepsilon} + \varepsilon, \quad P\text{-a.s.}, \end{aligned} \quad (3.14)$$

where $u^\varepsilon := u_1^\varepsilon \oplus u_2^\varepsilon \in \mathcal{U}_{t, T}$. From (3.12) and (3.14) and Lemmas 2.7 (comparison theorem for GBSDEs) and 2.6, we get

$$\begin{aligned} W_\delta(t, x) &\leq G_{t, t+\delta}^{t, x; u_1^\varepsilon} [Y_{t+\delta}^{t, x; u^\varepsilon} + \varepsilon] + \varepsilon \\ &\leq G_{t, t+\delta}^{t, x; u_1^\varepsilon} [Y_{t+\delta}^{t, x; u^\varepsilon}] + (C + 1)\varepsilon \\ &= G_{t, t+\delta}^{t, x; u^\varepsilon} [Y_{t+\delta}^{t, x; u^\varepsilon}] + (C + 1)\varepsilon \\ &= Y_t^{t, x; u^\varepsilon} + (C + 1)\varepsilon \\ &\leq \operatorname{esssup}_{u \in \mathcal{U}_{t, T}} Y_t^{t, x; u} + (C + 1)\varepsilon, \quad P\text{-a.s.} \end{aligned} \quad (3.15)$$

That is,

$$W_\delta(t, x) \leq W(t, x) + (C + 1)\varepsilon. \quad (3.16)$$

Finally, letting $\varepsilon \downarrow 0$, we get $W_\delta(t, x) \leq W(t, x)$.

Lemma 3.5. $W(t, x) \leq W_\delta(t, x)$, $0 \leq t < t + \delta \leq T$, $x \in \bar{D}$.

Proof. Since

$$W_\delta(t, x) = \operatorname{esssup}_{u_1 \in \mathcal{U}_{t, t+\delta}} I_\delta(t, x, u_1),$$

we have

$$\begin{aligned} W_\delta(t, x) &\geq I_\delta(t, x, u_1) \\ &= G_{t, t+\delta}^{t, x; u_1}[W(t + \delta, X_{t+\delta}^{t, x; u_1})], \quad P\text{-a.s., for all } u_1 \in \mathcal{U}_{t, t+\delta}. \end{aligned} \quad (3.17)$$

Moreover, from the definition of $W(t + \delta, y)$, $y \in \bar{D}$, we get

$$W(t + \delta, y) = \operatorname{esssup}_{u_2 \in \mathcal{U}_{t+\delta, T}} J(t + \delta, y; u_2), \quad P\text{-a.s.} \quad (3.18)$$

Let $\{O_i\}_{i \geq 1} \subset \mathcal{B}(\mathbb{R}^d)$ be a decomposition of \bar{D} such that $\sum_{i \geq 1} O_i = \bar{D}$ and $\operatorname{diam}(O_i) \leq \varepsilon$, $i \geq 1$. Let y_i be an arbitrarily given element of O_i , $i \geq 1$. Define

$$[X_{t+\delta}^{t, x; u_1}] := \sum_{i \geq 1} y_i \mathbf{1}_{\{X_{t+\delta}^{t, x; u_1} \in O_i\}}.$$

We have

$$|X_{t+\delta}^{t, x; u_1} - [X_{t+\delta}^{t, x; u_1}]| \leq \varepsilon, \quad \text{everywhere on } \Omega, \quad \text{for all } u_1 \in \mathcal{U}_{t, t+\delta}. \quad (3.19)$$

Let $u \in \mathcal{U}_{t, T}$ be arbitrarily given and decomposed into $u_1 = u|_{[t, t+\delta]} \in \mathcal{U}_{t, t+\delta}$ and $u_2 = u|_{(t+\delta, T]} \in \mathcal{U}_{t+\delta, T}$. Then, from (3.17), (3.13)(i), (3.19), and Lemmas 2.7 and 2.6, we have

$$\begin{aligned} W_\delta(t, x) &\geq G_{t, t+\delta}^{t, x; u_1}[W(t + \delta, X_{t+\delta}^{t, x; u_1})] \\ &\geq G_{t, t+\delta}^{t, x; u_1}[W(t + \delta, [X_{t+\delta}^{t, x; u_1}]) - C\varepsilon - C\varepsilon^{\frac{1}{2}}] - \varepsilon \\ &\geq G_{t, t+\delta}^{t, x; u_1}[W(t + \delta, [X_{t+\delta}^{t, x; u_1}])] - C\varepsilon - C'\varepsilon^{\frac{1}{2}} \\ &= G_{t, t+\delta}^{t, x; u_1} \left[\sum_{i \geq 1} \mathbf{1}_{\{X_{t+\delta}^{t, x; u_1} \in O_i\}} W(t + \delta, y_i) \right] - C\varepsilon - C'\varepsilon^{\frac{1}{2}}, \quad P\text{-a.s.} \end{aligned} \quad (3.20)$$

Furthermore, from (3.18), (3.13)(ii), (3.19), and Lemmas 2.7 and 2.6,

$$\begin{aligned} W_\delta(t, x) &\geq G_{t, t+\delta}^{t, x; u_1} \left[\sum_{i \geq 1} \mathbf{1}_{\{X_{t+\delta}^{t, x; u_1} \in O_i\}} J(t + \delta, y_i; u_2) \right] - C\varepsilon - C'\varepsilon^{\frac{1}{2}} \\ &= G_{t, t+\delta}^{t, x; u_1} [J(t + \delta, [X_{t+\delta}^{t, x; u_1}]; u_2)] - C\varepsilon - C'\varepsilon^{\frac{1}{2}} \\ &\geq G_{t, t+\delta}^{t, x; u_1} [J(t + \delta, X_{t+\delta}^{t, x; u_1}; u_2) - C''\varepsilon - C''\varepsilon^{\frac{1}{2}}] - C\varepsilon - C'\varepsilon^{\frac{1}{2}} \\ &\geq G_{t, t+\delta}^{t, x; u_1} [J(t + \delta, X_{t+\delta}^{t, x; u_1}; u_2)] - C\varepsilon - C'\varepsilon^{\frac{1}{2}} \\ &= G_{t, t+\delta}^{t, x; u} [Y_{t+\delta}^{t, x, u}] - C\varepsilon - C'\varepsilon^{\frac{1}{2}} \\ &= Y_t^{t, x; u} - C\varepsilon - C'\varepsilon^{\frac{1}{2}}, \quad P\text{-a.s., for any } u \in \mathcal{U}_{t, T}, \end{aligned} \quad (3.21)$$

where the constants C, C', C'' may vary from lines to lines. Consequently,

$$\begin{aligned} W_\delta(t, x) &\geq \operatorname{esssup}_{u \in \mathcal{U}_{t,T}} J(t, x; u) - C\varepsilon - C'\varepsilon^{\frac{1}{2}} \\ &= W(t, x) - C\varepsilon - C'\varepsilon^{\frac{1}{2}}, \quad P\text{-a.s.} \end{aligned} \quad (3.22)$$

Finally, letting $\varepsilon \downarrow 0$ we get $W_\delta(t, x) \geq W(t, x)$. The proof is complete.

Remark 3.2. (i) For any $u \in \mathcal{U}_{t,t+\delta}$,

$$W(t, x)(= W_\delta(t, x)) \geq G_{t,t+\delta}^{t,x;u}[W(t+\delta, X_{t+\delta}^{t,x;u})], \quad P\text{-a.s.} \quad (3.23)$$

(ii) From the inequality (3.12), for all $(t, x) \in [0, T] \times \mathbb{R}^n$, $\delta \in (0, T-t]$ and $\varepsilon > 0$, the following holds: there exists some $u^\varepsilon(\cdot) \in \mathcal{U}_{t,t+\delta}$ such that

$$W(t, x)(= W_\delta(t, x)) \leq G_{t,t+\delta}^{t,x;u^\varepsilon}[W(t+\delta, X_{t+\delta}^{t,x;u^\varepsilon})] + C\varepsilon, \quad P\text{-a.s.} \quad (3.24)$$

(iii) Recall that the value function W is deterministic. Then, with $\delta = T-t$ and taking the expectation on both sides of (3.23) and (3.24) we can get that

$$W(t, x) = \sup_{u \in \mathcal{U}_{t,T}} E[J(t, x; u)].$$

Lemma 3.2 shows that the value function $W(t, x)$ is continuous in x , uniformly in t . From Theorem 3.1 we can get the continuity of $W(t, x)$ in t .

Theorem 3.2. Let assumptions (H3.1) and (H3.2) be satisfied. Then the value function $W(t, x)$ is continuous in t .

Proof. Let $(t, x) \in [0, T] \times \bar{D}$ and $\delta \in (0, T-t]$. We want to prove that W is continuous in t . For this we notice that from (3.24), for an arbitrarily small $\varepsilon > 0$,

$$I_\delta^1 + I_\delta^2 \leq W(t, x) - W(t+\delta, x) \leq I_\delta^1 + I_\delta^2 + C\varepsilon, \quad (3.25)$$

where

$$\begin{aligned} I_\delta^1 &:= G_{t,t+\delta}^{t,x;u^\varepsilon}[W(t+\delta, X_{t+\delta}^{t,x;u^\varepsilon})] - G_{t,t+\delta}^{t,x;u^\varepsilon}[W(t+\delta, x)], \\ I_\delta^2 &:= G_{t,t+\delta}^{t,x;u^\varepsilon}[W(t+\delta, x)] - W(t+\delta, x) \end{aligned}$$

for $u^\varepsilon \in \mathcal{U}_{t,t+\delta}$ such that (3.24) holds. From Lemma 2.6 and the estimate (3.8) we get that, for some constant C which does not depend on the controls u^ε ,

$$\begin{aligned} |I_\delta^1| &\leq [CE(|W(t+\delta, X_{t+\delta}^{t,x;u^\varepsilon}) - W(t+\delta, x)|^2 | \mathcal{F}_t)]^{\frac{1}{2}} \\ &\leq [CE(|X_{t+\delta}^{t,x;u^\varepsilon} - x|^2 + |X_{t+\delta}^{t,x;u^\varepsilon} - x| | \mathcal{F}_t)]^{\frac{1}{2}}, \end{aligned}$$

and since $E[|X_{t+\delta}^{t,x;u^\varepsilon} - x|^2 | \mathcal{F}_t] \leq C\delta$ (refer to (5.15) in the appendix) we get that $|I_\delta^1| \leq C\delta^{\frac{1}{4}}$. From the definition of $G_{t,t+\delta}^{t,x;u^\varepsilon}[\cdot]$ (see (3.9)),

$$\begin{aligned} I_\delta^2 &= E \left[W(t+\delta, x) + \int_t^{t+\delta} f(s, X_s^{t,x;u^\varepsilon}, \tilde{Y}_s^{t,x;u^\varepsilon}, \tilde{Z}_s^{t,x;u^\varepsilon}, u_s^\varepsilon) ds \right. \\ &\quad \left. + \int_t^{t+\delta} g(s, X_s^{t,x;u^\varepsilon}, \tilde{Y}_s^{t,x;u^\varepsilon}) dK_s^{t,x;u^\varepsilon} - \int_t^{t+\delta} \tilde{Z}_s^{t,x;u^\varepsilon} dB_s | \mathcal{F}_t \right] - W(t+\delta, x) \\ &= E \left[\int_t^{t+\delta} f(s, X_s^{t,x;u^\varepsilon}, \tilde{Y}_s^{t,x;u^\varepsilon}, \tilde{Z}_s^{t,x;u^\varepsilon}, u_s^\varepsilon) ds + \int_t^{t+\delta} g(s, X_s^{t,x;u^\varepsilon}, \tilde{Y}_s^{t,x;u^\varepsilon}) dK_s^{t,x;u^\varepsilon} | \mathcal{F}_t \right]. \end{aligned}$$

From the Schwartz inequality, Propositions 5.2 and 5.3 in the Appendix and (3.2), we then get

$$\begin{aligned}
|I_\delta^2| &\leq \delta^{\frac{1}{2}} E \left[\int_t^{t+\delta} |f(s, X_s^{t,x;u^\varepsilon}, \tilde{Y}_s^{t,x;u^\varepsilon}, \tilde{Z}_s^{t,x;u^\varepsilon}, u_s^\varepsilon)|^2 ds | \mathcal{F}_t \right]^{\frac{1}{2}} \\
&\quad + E \left[K_{t+\delta}^{t,x;u^\varepsilon} | \mathcal{F}_t \right]^{\frac{1}{2}} E \left[\int_t^{t+\delta} |g(s, X_s^{t,x;u^\varepsilon}, \tilde{Y}_s^{t,x;u^\varepsilon})|^2 dK_s^{t,x;u^\varepsilon} | \mathcal{F}_t \right]^{\frac{1}{2}} \\
&\leq \delta^{\frac{1}{2}} E \left[\int_t^{t+\delta} (|f(s, X_s^{t,x;u^\varepsilon}, 0, 0, u_s^\varepsilon)| + C|\tilde{Y}_s^{t,x;u^\varepsilon}| + C|\tilde{Z}_s^{t,x;u^\varepsilon}|)^2 ds | \mathcal{F}_t \right]^{\frac{1}{2}} \\
&\quad + E \left[K_{t+\delta}^{t,x;u^\varepsilon} | \mathcal{F}_t \right]^{\frac{1}{2}} E \left[\int_t^{t+\delta} (|g(s, X_s^{t,x;u^\varepsilon}, 0)| + C|\tilde{Y}_s^{t,x;u^\varepsilon}|)^2 dK_s^{t,x;u^\varepsilon} | \mathcal{F}_t \right]^{\frac{1}{2}} \\
&\leq C\delta^{\frac{1}{2}} E \left[\int_t^{t+\delta} (1 + |X_s^{t,x;u^\varepsilon}| + |\tilde{Y}_s^{t,x;u^\varepsilon}| + |\tilde{Z}_s^{t,x;u^\varepsilon}|)^2 ds | \mathcal{F}_t \right]^{\frac{1}{2}} \\
&\quad + CE \left[K_{t+\delta}^{t,x;u^\varepsilon} | \mathcal{F}_t \right]^{\frac{1}{2}} E \left[\int_t^{t+\delta} (1 + |X_s^{t,x;u^\varepsilon}| + |\tilde{Y}_s^{t,x;u^\varepsilon}|)^2 dK_s^{t,x;u^\varepsilon} | \mathcal{F}_t \right]^{\frac{1}{2}} \\
&\leq C\delta^{\frac{1}{2}} + C \left(E \left[|K_{t+\delta}^{t,x;u^\varepsilon}|^2 | \mathcal{F}_t \right] \right)^{\frac{1}{2}} \leq C\delta^{\frac{1}{2}}.
\end{aligned}$$

Then, from (3.25),

$$|W(t, x) - W(t + \delta, x)| \leq C\delta^{\frac{1}{4}} + C\delta^{\frac{1}{2}} + C\varepsilon,$$

and letting $\varepsilon \downarrow 0$ we get $W(t, x)$ is continuous in t . The proof is complete.

4 Viscosity solutions of related HJB equations

We consider the following PDE:

$$\begin{cases} \frac{\partial}{\partial t} W(t, x) + H(t, x, W, DW, D^2W) = 0, & (t, x) \in [0, T] \times D, \\ \frac{\partial}{\partial n} W(t, x) + g(t, x, W(t, x)) = 0, & 0 \leq t < T, \ x \in \partial D; \\ W(T, x) = \Phi(x), & x \in \bar{D}, \end{cases} \quad (4.1)$$

where at a point $x \in \partial D$, $\frac{\partial}{\partial n} = \sum_{i=1}^d \frac{\partial}{\partial x_i} \phi(x) \frac{\partial}{\partial x_i}$, and the Hamiltonian

$$H(t, x, y, p, X) = \sup_{u \in U} \left\{ \frac{1}{2} \text{tr}(\sigma \sigma^T(t, x, u) X) + p \cdot b(t, x, u) + f(t, x, y, p\sigma, u) \right\},$$

where $t \in [0, T]$, $x \in \mathbb{R}^n$, $y \in \mathbb{R}$, $p \in \mathbb{R}^d$, and $X \in \mathbf{S}^d$ (recall that \mathbf{S}^d denotes the set of $d \times d$ symmetric matrices). Here the two pairs of functions (b, σ) and (f, Φ) are supposed to satisfy assumptions (H3.1) and (H3.2), respectively.

In this section we shall prove that the value function W defined by (3.7) is a viscosity solution of (4.1). The interested reader is referred to Crandall, Ishii, and Lions [5] for a detailed introduction to viscosity solutions. Let $C_{l,b}^3([0, T] \times \bar{D})$ be the set of the real-valued functions that are continuously differentiable up to the third order and whose derivatives of order from 1 to 3 are bounded.

Definition 4.1. A real-valued continuous function $W \in C([0, T] \times \bar{D})$ is called

(i) a viscosity subsolution of (4.1) if $W(T, x) \leq \Phi(x)$ for all $x \in \bar{D}$ and if for all functions $\varphi \in C_{l,b}^3([0, T] \times \bar{D})$ and $(t, x) \in [0, T] \times \bar{D}$ such that $W - \varphi$ attains its local maximum at (t, x) :

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(t, x) + H(t, x, W, D\varphi, D^2\varphi) &\geq 0, \quad \text{if } x \in D; \\ \max \left\{ \frac{\partial \varphi}{\partial t}(t, x) + H(t, x, W, D\varphi, D^2\varphi), \quad \frac{\partial \varphi}{\partial n}(t, x) + g(t, x, W) \right\} &\geq 0, \quad \text{if } x \in \partial D; \end{aligned}$$

(ii) a viscosity supersolution of (4.1) if $W(T, x) \geq \Phi(x)$ for all $x \in \bar{D}$ and if for all functions $\varphi \in C_{l,b}^3([0, T] \times \bar{D})$ and $(t, x) \in [0, T] \times \bar{D}$ such that $W - \varphi$ attains its local minimum at (t, x) :

$$\begin{aligned} \frac{\partial \varphi}{\partial t}(t, x) + H(t, x, W, D\varphi, D^2\varphi) &\leq 0, \quad \text{if } x \in D; \\ \min \left\{ \frac{\partial \varphi}{\partial t}(t, x) + H(t, x, W, D\varphi, D^2\varphi), \quad \frac{\partial \varphi}{\partial n}(t, x) + g(t, x, W) \right\} &\leq 0, \quad \text{if } x \in \partial D; \end{aligned}$$

(iii) a viscosity solution of (4.1) if it is both a viscosity sub- and a supersolution of (4.1).

For simplicity of notation, define for $\varphi \in C_{l,b}^3([0, T] \times \bar{D})$,

$$\begin{aligned} F(s, x, y, z, u) &= \frac{\partial}{\partial s} \varphi(s, x) + \frac{1}{2} \text{tr}(\sigma \sigma^T(s, x, u) D^2 \varphi) + D\varphi \cdot b(s, x, u) \\ &\quad + f(s, x, y + \varphi(s, x), z + D\varphi(s, x) \cdot \sigma(s, x, u), u), \\ G(s, x, y) &= \frac{\partial}{\partial n} \varphi(s, x) + g(s, x, y + \varphi(s, x)) \end{aligned} \quad (4.2)$$

for $(s, x, y, z, u) \in [0, T] \times \bar{D} \times \mathbb{R} \times \mathbb{R}^d \times U$.

Proposition 4.1. Under the assumptions (H3.1) and (H3.2) the value function W is a viscosity subsolution of (4.1).

Proof. Obviously, $W(T, x) = \Phi(x)$, $x \in \bar{D}$. Suppose that $\varphi \in C_{l,b}^3([0, T] \times \bar{D})$ and $(t, x) \in [0, T] \times \bar{D}$ is such that $W - \varphi$ attains its maximum at (t, x) . Without loss of generality, assume that $\varphi(t, x) = W(t, x)$.

We first consider the case $x \in D$. We shall prove that

$$\sup_{u \in U} F(t, x, 0, 0, u) \geq 0.$$

If this is not true, then there exists some $\theta > 0$ such that

$$F_0(t, x) := \sup_{u \in U} F(t, x, 0, 0, u) \leq -\theta < 0. \quad (4.3)$$

Therefore,

$$F(t, x, 0, 0, u) \leq -\theta \text{ for all } u \in U.$$

Since F_0 is continuous at (t, x) , we can choose $\bar{\alpha} \in (0, T - t]$ such that

$$O_{\bar{\alpha}}(x) := \{y : |y - x| \leq \bar{\alpha}\} \subset D, \quad (4.4)$$

$$F(s, y, 0, 0, u) \leq -\frac{1}{2}\theta \text{ for all } (s, y, u) \in [t, t + \bar{\alpha}] \times O_{\bar{\alpha}}(x) \times U. \quad (4.5)$$

For any $\alpha \in (0, \bar{\alpha}]$, consider the following BSDE:

$$\begin{cases} -dY_s^{1,u} &= F(s, X_s^{t,x;u}, Y_s^{1,u}, Z_s^{1,u}, u_s) ds + G(s, X_s^{t,x;u}, Y_s^{1,u}) dK_s^{t,x;u} \\ &\quad - Z_s^{1,u} dB_s, \quad s \in [t, t+\alpha]; \\ Y_{t+\alpha}^{1,u} &= 0, \end{cases} \quad (4.6)$$

where the pair of processes $(X^{t,x;u}, K^{t,x;u})$ are given by (3.1) and $u(\cdot) \in \mathcal{U}_{t,t+\alpha}$. It is not hard to check that $F(s, X_s^{t,x;u}, y, z, u_s)$ and $G(s, X_s^{t,x;u}, y)$ satisfy (H2.1). Thus, due to Lemma 2.4, GBSDE (4.6) has a unique solution. We have the following observation.

Lemma 4.1. *For every $s \in [t, t+\alpha]$, we have the following relationship:*

$$Y_s^{1,u} = G_{s,t+\alpha}^{t,x;u}[\varphi(t+\alpha, X_{t+\alpha}^{t,x;u})] - \varphi(s, X_s^{t,x;u}), \quad P\text{-a.s.} \quad (4.7)$$

Proof. We recall that $G_{s,t+\alpha}^{t,x;u}[\varphi(t+\alpha, X_{t+\alpha}^{t,x;u})]$ is defined by the solution of the GBSDE

$$\begin{cases} -dY_s^u &= f(s, X_s^{t,x;u}, Y_s^u, Z_s^u, u_s) ds + g(s, X_s^{t,x;u}, Y_s^u) dK_s^{t,x;u} \\ &\quad - Z_s^u dB_s, \quad s \in [t, t+\alpha]; \\ Y_{t+\alpha}^u &= \varphi(t+\alpha, X_{t+\alpha}^{t,x;u}), \end{cases}$$

with the following formula:

$$G_{s,t+\alpha}^{t,x;u}[\varphi(t+\alpha, X_{t+\alpha}^{t,x;u})] = Y_s^u, \quad s \in [t, t+\alpha], \quad (4.8)$$

(see (3.9)). Hence, we only need to show that $Y_s^u - \varphi(s, X_s^{t,x;u}) \equiv Y_s^{1,u}$ for $s \in [t, t+\alpha]$. This can be verified directly by applying Itô's formula to $\varphi(s, X_s^{t,x;u})$. Indeed, the stochastic differentials of $Y_s^u - \varphi(s, X_s^{t,x;u})$ and $Y_s^{1,u}$ equal, and with the same terminal condition $Y_{t+\alpha}^u - \varphi(t+\alpha, X_{t+\alpha}^{t,x;u}) = 0 = Y_{t+\alpha}^{1,u}$.

Remark 4.1. *For $x \in \partial D$ Lemma 4.1 still holds.*

On the other hand, from the DPP (see Theorem 3.1), for every α ,

$$\varphi(t, x) = W(t, x) = \operatorname{esssup}_{u \in \mathcal{U}_{t,t+\alpha}} G_{t,t+\alpha}^{t,x;u}[W(t+\alpha, X_{t+\alpha}^{t,x;u})],$$

and from $W \leq \varphi$ and the monotonicity property of $G_{t,t+\delta}^{t,x;u}[\cdot]$ (see Lemma 2.7) we get

$$\operatorname{esssup}_{u \in \mathcal{U}_{t,t+\alpha}} \{G_{t,t+\delta}^{t,x;u}[\varphi(t+\alpha, X_{t+\alpha}^{t,x;u})] - \varphi(t, x)\} \geq 0, \quad P\text{-a.s.}$$

Thus, from Lemma 4.1, we have

$$\operatorname{esssup}_{u \in \mathcal{U}_{t,t+\alpha}} Y_t^{1,u} \geq 0, \quad P\text{-a.s.}$$

Hence, given an arbitrary $\varepsilon > 0$ we can choose $u^\varepsilon \in \mathcal{U}_{t,t+\alpha}$ such that, P-a.s.,

$$Y_t^{1,u^\varepsilon} \geq -\varepsilon\alpha. \quad (4.9)$$

(The proof is similar to that of inequality (3.24).)

Remark 4.2. Similarly, for $x \in \partial D$ we also have (4.9).

For $u^\varepsilon \in \mathcal{U}_{t,t+\alpha}$ define

$$\tau = \inf \{s \geq t : |X_s^{t,x;u^\varepsilon} - x| \geq \bar{\alpha}\} \wedge (t + \alpha).$$

Consequently, on $[t, \tau]$ the process $(K^{t,x;u})$ is zero and, hence

$$Y_s^{1;u^\varepsilon} = Y_\tau^{1;u^\varepsilon} + \int_s^\tau F(r, X_r^{t,x;u^\varepsilon}, Y_r^{1;u^\varepsilon}, Z_r^{1;u^\varepsilon}, u_r^\varepsilon) dr - \int_s^\tau Z_r^{1;u^\varepsilon} dB_r.$$

Consider the following two BSDEs:

$$\begin{cases} -dY_s^2 &= [C^*(|Y_s^2| + |Z_s^2|) - \frac{1}{2}\theta] ds - Z_s^2 dB_s, \\ Y_{t+\alpha}^2 &= 0 \end{cases} \quad (4.10)$$

whose unique solution is given by

$$Y_s^2 = -\frac{\theta}{2C^*} \left(1 - e^{C^*(s-(t+\alpha))}\right), \quad Z_s^2 = 0, \quad s \in [t, t + \alpha], \quad (4.11)$$

and

$$\begin{cases} -dY_s^3 &= [C^*(|Y_s^3| + |Z_s^3|) - \frac{1}{2}\theta] ds - Z_s^3 dB_s, \quad s \in [t, \tau]; \\ Y_\tau^3 &= Y_\tau^{1;u^\varepsilon}. \end{cases} \quad (4.12)$$

Here, C^* is the Lipschitz constant of F with respect to y, z ; also the Lipschitz constant of G with respect to y , in order to be different from the constant C which may vary from lines to lines. We have the following lemma.

Lemma 4.2. We have $Y_t^{1,u^\varepsilon} \leq Y_t^3$ and $|Y_t^2 - Y_t^3| \leq C\alpha^{\frac{3}{2}}, P$ -a.s. Here $C > 0$ is independent of both the control u and α .

Proof. (1) We observe from (4.5) and the definition of τ that, for all $(s, y, z, u) \in [t, \tau] \times \mathbb{R} \times \mathbb{R}^d \times U$,

$$\begin{aligned} F(s, X_s^{t,x;u^\varepsilon}, y, z, u^\varepsilon) &\leq C^*(|y| + |z|) + F(s, X_s^{t,x;u^\varepsilon}, 0, 0, u^\varepsilon) \\ &\leq C^*(|y| + |z|) - \frac{1}{2}\theta. \end{aligned}$$

Consequently, from Lemma 2.2 (the comparison result for BSDEs) we have that

$$Y_s^{1,u^\varepsilon} \leq Y_s^3, \quad s \in [t, \tau], \quad P\text{-a.s.},$$

where Y^3 is defined by BSDE (4.12).

(2) From the equation (4.6), Proposition 5.1 and Proposition 5.2 in the Appendix, we have

$$|Y_\tau^{1;u^\varepsilon}| \leq C(t + \alpha - \tau)^{\frac{1}{2}} + C \left(E \left[(K_{t+\alpha}^{t,x;u^\varepsilon} - K_\tau^{t,x;u^\varepsilon})^2 \mid \mathcal{F}_\tau \right] \right)^{\frac{1}{2}},$$

where C is independent of controls, and $K_{t+\alpha}^{t,x;u^\varepsilon} - K_\tau^{t,x;u^\varepsilon} = K_{t+\alpha}^{\tau, X_\tau^{t,x;u^\varepsilon}; u^\varepsilon}$ by means of the uniqueness of solution of reflected SDE (3.1). Therefore, we have

$$E \left[|Y_\tau^{1;u^\varepsilon}|^2 \mid \mathcal{F}_t \right] \leq CE \left[(t + \alpha - \tau) \mid \mathcal{F}_t \right] + CE \left[\left| K_{t+\alpha}^{\tau, X_\tau^{t,x;u^\varepsilon}; u^\varepsilon} \right|^2 \mid \mathcal{F}_t \right].$$

From Proposition 5.3 in the Appendix, we have

$$E \left[\left| K_{t+\alpha}^{\tau, X_{\tau}^{t,x;u^\varepsilon}; u^\varepsilon} \right|^2 \mid \mathcal{F}_t \right] \leq C (E [(t + \alpha - \tau)^2 \mid \mathcal{F}_t])^{\frac{1}{2}}. \quad (4.13)$$

Therefore, we get

$$E \left[\left| Y_{\tau}^{1;u^\varepsilon} \right|^2 \mid \mathcal{F}_t \right] \leq C (E [(t + \alpha - \tau)^2 \mid \mathcal{F}_t])^{\frac{1}{2}}. \quad (4.14)$$

On the other hand, we consider the following SDE:

$$d\bar{X}_s^{t,x;u^\varepsilon} = b(s, \bar{X}_s^{t,x;u^\varepsilon}, u_s^\varepsilon) ds + \sigma(s, \bar{X}_s^{t,x;u^\varepsilon}, u_s^\varepsilon) dB_s, \quad s \geq t; \quad \bar{X}_t^{t,x;u^\varepsilon} = x. \quad (4.15)$$

Then we know on $[t, \tau]$, P-a.s.,

$$X^{t,x;u^\varepsilon} = \bar{X}^{t,x;u^\varepsilon}.$$

For $\bar{X}^{t,x;u^\varepsilon}$ we have the classical estimate

$$E \left[\sup_{t \leq s \leq t+\alpha} |\bar{X}_s^{t,x;u^\varepsilon} - x|^8 \mid \mathcal{F}_t \right] \leq C\alpha^4, \quad \text{P-a.s.}$$

Therefore, we have

$$P \{ \tau < t + \alpha \mid \mathcal{F}_t \} \leq P \left\{ \sup_{s \in [t, t+\alpha]} |\bar{X}_s^{t,x;u^\varepsilon} - x| \geq \bar{\alpha} \mid \mathcal{F}_t \right\} \leq \frac{C}{\bar{\alpha}^8} \alpha^4. \quad (4.16)$$

Hence,

$$E \left[\left| Y_{\tau}^{1;u^\varepsilon} \right|^2 \mid \mathcal{F}_t \right] \leq C\alpha (P \{ \tau < t + \alpha \mid \mathcal{F}_t \})^{\frac{1}{2}} \leq \frac{C}{\bar{\alpha}^4} \alpha^3. \quad (4.17)$$

Furthermore, from Lemma 2.3

$$\begin{aligned} |Y_t^2 - Y_t^3| &\leq C (E[|Y_\tau^2 - Y_\tau^3|^2 \mid \mathcal{F}_t])^{\frac{1}{2}} \\ &\leq C (E[|Y_\tau^2|^2 \mid \mathcal{F}_t])^{\frac{1}{2}} + C (E[|Y_\tau^3|^2 \mid \mathcal{F}_t])^{\frac{1}{2}} \\ &\leq C \frac{\theta}{2} (1 - e^{-C^* \alpha}) (P \{ \tau < t + \alpha \mid \mathcal{F}_t \})^{\frac{1}{2}} + C (E[|Y_\tau^{1;u^\varepsilon}|^2 \mid \mathcal{F}_t])^{\frac{1}{2}} \\ &\leq C \frac{\theta}{2} (1 - e^{-C^* \alpha}) \frac{1}{\bar{\alpha}^4} \alpha^2 + \frac{C}{\bar{\alpha}^2} \alpha^{\frac{3}{2}} \\ &\leq C \alpha^{\frac{3}{2}}, \end{aligned} \quad (4.18)$$

for any $\alpha \in (0, \bar{\alpha}]$.

The above auxiliary results now allow to complete the proof of Proposition 4.1.

Proof of Proposition 4.1 (sequel).

By combining (4.9) with Lemma 4.2 we then obtain

$$-\varepsilon\alpha \leq Y_t^{1,u^\varepsilon} \leq Y_t^3 \leq Y_t^2 + |Y_t^2 - Y_t^3| \leq Y_t^2 + C\alpha^{\frac{3}{2}}, \quad \text{P-a.s.}$$

i.e.,

$$-\varepsilon\alpha \leq Y_t^{1,u^\varepsilon} \leq -\frac{\theta}{2C^*} \left(1 - e^{-C^*\alpha}\right) + C\alpha^{\frac{3}{2}}, \text{ P-a.s.}$$

Therefore,

$$-\varepsilon \leq -\frac{\theta}{2C^*} \frac{1 - e^{-C^*\alpha}}{\alpha} + C\alpha^{\frac{1}{2}}.$$

Letting $\alpha \rightarrow 0+$ and $\varepsilon \rightarrow 0+$, we get $0 \leq -\frac{\theta}{2}$, which contradicts our assumption that $\theta > 0$. Therefore, we have

$$\sup_{u \in U} F(t, x, 0, 0, u) \geq 0,$$

which implies by the definition of F that

$$\frac{\partial \varphi}{\partial t}(t, x) + H(t, x, W, D\varphi, D^2\varphi) \geq 0, \quad \text{if } x \in D.$$

We now consider the case $x \in \partial D$. We must prove that

$$\max \left\{ \frac{\partial \varphi}{\partial t}(t, x) + H(t, x, \varphi, D\varphi, D^2\varphi), \quad \frac{\partial \varphi}{\partial n}(t, x) + g(t, x, \varphi) \right\} \geq 0$$

If this is not true, then there exists some $\theta > 0$ such that

$$\sup_{u \in U} F(t, x, 0, 0, u) \leq -\theta < 0, \quad G(t, x, 0) \leq -\theta < 0, \quad (4.19)$$

therefore,

$$F(t, x, 0, 0, u) \leq -\theta \text{ for all } u \in U.$$

$$G(t, x, 0) \leq -\theta \text{ for all } u \in U.$$

Choose $\bar{\alpha} \in (0, T - t]$ such that

$$F(s, y, 0, 0, u) \leq -\frac{1}{2}\theta, \quad (4.20)$$

$$G(s, y, 0) \leq -\frac{1}{2}\theta, \text{ for all } u \in U, t \leq s \leq t + \bar{\alpha}, |y - x| \leq \bar{\alpha}. \quad (4.21)$$

Now we fix $\bar{\alpha}$, and we consider any $\alpha \in (0, \bar{\alpha}]$. Similarly, we consider GBSDE (4.6) with $x \in \partial D$, then we also can get (4.7) and (4.9). For $u^\varepsilon \in \mathcal{U}_{t,t+\alpha}$ in (4.9) define

$$\tau = \inf \{s \geq t : |X_s^{t,x;u^\varepsilon} - x| \geq \bar{\alpha}\} \wedge (t + \alpha);$$

We observe that, for all $(s, y, z) \in [t, \tau] \times \mathbb{R} \times \mathbb{R}^d$, from (4.20), (4.21) and the definition of τ

$$\begin{aligned} F(s, X_s^{t,x;u^\varepsilon}, y, z, u_s^\varepsilon) &\leq C^*(|y| + |z|) + F(s, X_s^{t,x;u^\varepsilon}, 0, 0, u_s^\varepsilon) \\ &\leq C^*(|y| + |z|) - \frac{1}{2}\theta. \end{aligned}$$

$$\begin{aligned} G(s, X_s^{t,x;u^\varepsilon}, y) &\leq C^*|y| + G(s, X_s^{t,x;u^\varepsilon}, 0) \\ &\leq C^*|y| - \frac{1}{2}\theta. \end{aligned}$$

Consequently, applying the comparison result for GBSDEs (Lemma 2.7, or Remark 1.5 in Pardoux and Zhang [17]) to GBSDEs (4.6) and (4.23) we have that

$$Y_s^{1,u^\varepsilon} \leq Y_s^4, \quad s \in [t, \tau], \quad \text{P-a.s.}, \quad (4.22)$$

where Y^4 is defined by the following BSDE:

$$\begin{cases} -dY_s^4 = \{C^*(|Y_s^4| + |Z_s^4|) - \frac{1}{2}\theta\}ds + (C^*|Y_s^4| - \frac{1}{2}\theta)dK_s^{t,x;u^\varepsilon} - Z_s^4dB_s, \\ Y_\tau^4 = Y_\tau^{1;u^\varepsilon}. \end{cases} \quad (4.23)$$

On the other hand, we also have to introduce the following BSDE:

$$\begin{cases} -dY_s^5 = \{C^*(|Y_s^5| + |Z_s^5|) - \frac{1}{2}\theta\}ds + (C^*|Y_s^5| - \frac{1}{2}\theta)dK_s^{t,x;u^\varepsilon} - Z_s^5dB_s, \\ Y_{t+\alpha}^5 = 0. \end{cases} \quad (4.24)$$

Notice that $C^*|Y_s^2| - \frac{1}{2}\theta < 0$, therefore $Y_s^5 \leq Y_s^2, s \in [t, t+\alpha]$, P-a.s., from the comparison theorem-Lemma 2.7.

From Lemma 2.6 we have

$$\begin{aligned} |Y_t^4 - Y_t^5| &\leq C(E[|Y_\tau^4 - Y_\tau^5|^2|\mathcal{F}_t])^{\frac{1}{2}} \\ &\leq C(E[|Y_\tau^4|^2|\mathcal{F}_t])^{\frac{1}{2}} + C(E[|Y_\tau^5|^2|\mathcal{F}_t])^{\frac{1}{2}} \\ &\leq C(E[|Y_\tau^{1;u^\varepsilon}|^2|\mathcal{F}_t])^{\frac{1}{2}} + C(E[|Y_\tau^2|^2|\mathcal{F}_t])^{\frac{1}{2}} + C(E[|Y_\tau^5 - Y_\tau^2|^2|\mathcal{F}_t])^{\frac{1}{2}} \\ &\leq C\alpha^{\frac{3}{2}} + C(E[|Y_\tau^5 - Y_\tau^2|^2|\mathcal{F}_t])^{\frac{1}{2}} \quad (\text{from the proof of (4.18)}), \end{aligned}$$

for any $\alpha \in (0, \bar{\alpha}]$. From (5.16) of Remark 5.3 in the Appendix, similarly we also have

$$P\{\tau < t + \alpha | \mathcal{F}_t\} \leq P\left\{\sup_{s \in [t, t+\alpha]} |X_s^{t,x;u^\varepsilon} - x| \geq \bar{\alpha} \mid \mathcal{F}_t\right\} \leq \frac{C}{\bar{\alpha}^8} \alpha^4. \quad (4.25)$$

On the other hand, from Lemma 2.6 (taking $\mu = 1$)

$$\begin{aligned} &E[|Y_\tau^5 - Y_\tau^2|^2|\mathcal{F}_t] \\ &\leq CE[\int_\tau^{t+\alpha} e^{2K_s^{t,x;u^\varepsilon}} (C^*|Y_s^2| - \frac{1}{2}\theta)^2 dK_s^{t,x;u^\varepsilon} | \mathcal{F}_t] \\ &= CE[\int_\tau^{t+\alpha} e^{2K_s^{t,x;u^\varepsilon}} \frac{\theta^2}{4} e^{2C^*(s-(t+\alpha))} dK_s^{t,x;u^\varepsilon} | \mathcal{F}_t] \\ &\leq C\frac{\theta^2}{4} E[I_{\{\tau < t+\alpha\}} (e^{2K_{t+\alpha}^{t,x;u^\varepsilon}} - e^{2K_\tau^{t,x;u^\varepsilon}}) | \mathcal{F}_t] \\ &\leq C\frac{\theta^2}{4} E[I_{\{\tau < t+\alpha\}} e^{2K_{t+\alpha}^{t,x;u^\varepsilon}} (K_{t+\alpha}^{t,x;u^\varepsilon} - K_\tau^{t,x;u^\varepsilon}) | \mathcal{F}_t] \\ &\leq C\frac{\theta^2}{4} \left(P\{\tau < t + \alpha | \mathcal{F}_t\}\right)^{\frac{1}{4}} \left(E[e^{8K_{t+\alpha}^{t,x;u^\varepsilon}} | \mathcal{F}_t]\right)^{\frac{1}{4}} (E[|K_{t+\alpha}^{\tau, X_\tau^{t,x;u^\varepsilon}; u^\varepsilon}|^2 | \mathcal{F}_t])^{\frac{1}{2}} \\ &\leq C\frac{\theta^2}{4} (P\{\tau < t + \alpha | \mathcal{F}_t\})^{\frac{1}{4}} (E[(t + \alpha - \tau)^2 | \mathcal{F}_t])^{\frac{1}{4}} \quad (\text{from Prop. 5.1 and 5.3.}) \\ &\leq C\theta^2 (P\{\tau < t + \alpha | \mathcal{F}_t\})^{\frac{1}{4}} (\alpha^2 P\{\tau < t + \alpha | \mathcal{F}_t\})^{\frac{1}{4}} \\ &\leq C\theta^2 \alpha^{\frac{5}{2}}. \end{aligned} \quad (4.26)$$

Therefore,

$$|Y_t^4 - Y_t^5| \leq C\alpha^{\frac{3}{2}} + C\theta\alpha^{\frac{5}{4}}. \quad (4.27)$$

Now we obtain

$$-\varepsilon\alpha \leq Y_t^{1;u^\varepsilon} \leq Y_t^4 \leq Y_t^5 + |Y_t^4 - Y_t^5| \leq Y_t^2 + C\alpha^{\frac{3}{2}} + C\theta\alpha^{\frac{5}{4}}, \text{P-a.s.}$$

i.e.,

$$-\varepsilon\alpha \leq Y_t^{1;u^\varepsilon} \leq -\frac{\theta}{2C^*} (1 - e^{-C^*\alpha}) + C\alpha^{\frac{3}{2}} + C\theta\alpha^{\frac{5}{4}}, \text{P-a.s.}$$

Therefore,

$$-\varepsilon \leq -\frac{\theta}{2C^*} \frac{1 - e^{-C^*\alpha}}{\alpha} + C\alpha^{\frac{1}{2}} + C\theta\alpha^{\frac{1}{4}},$$

and by taking the limit as $\alpha \downarrow 0, \varepsilon \downarrow 0$ we get $0 \leq -\frac{\theta}{2}$ which contradicts our assumption that $\theta > 0$. Therefore, it must hold

$$\max \left\{ \frac{\partial \varphi}{\partial t}(t, x) + H(t, x, W, D\varphi, D^2\varphi), \quad \frac{\partial \varphi}{\partial n}(t, x) + g(t, x, W) \right\} \geq 0.$$

□

In an identical way, we can show

Proposition 4.2. *Under the assumptions (H4.1) and (H4.2), the value function W is a viscosity supersolution to (4.1).*

Proof. Obviously, $W(T, x) = \Phi(x)$, $x \in \bar{D}$. Suppose that $\varphi \in C_{l,b}^3([0, T] \times \bar{D})$ and $(t, x) \in [0, T) \times \bar{D}$ is such that $W - \varphi$ attains its minimum at (t, x) . Without loss of generality, assume that $\varphi(t, x) = W(t, x)$.

We first consider the case $x \in D$. We shall prove that

$$\sup_{u \in U} F(t, x, 0, 0, u) \leq 0.$$

If this is not true, then there exists some $\theta > 0$ such that

$$F_0(t, x) := \sup_{u \in U} F(t, x, 0, 0, u) \geq \theta > 0. \quad (4.28)$$

Therefore, there exists a $u^* = u^*(t, x) \in U$ such that

$$F(t, x, 0, 0, u^*) \geq \frac{2\theta}{3}.$$

Since F_0 is continuous at (t, x) , we can choose $\bar{\alpha} \in (0, T - t]$ (for simplifying the notation, we still use $\bar{\alpha}$) such that

$$O_{\bar{\alpha}}(x) := \{y : |y - x| \leq \bar{\alpha}\} \subset D, \quad (4.29)$$

$$F(s, y, 0, 0, u^*) \geq \frac{1}{2}\theta \text{ for all } (s, y) \in [t, t + \bar{\alpha}] \times O_{\bar{\alpha}}(x). \quad (4.30)$$

For any $\alpha \in (0, \bar{\alpha}]$, we still consider the BSDE (4.6):

$$\begin{cases} -dY_s^{1,u} &= F(s, X_s^{t,x,u}, Y_s^{1,u}, Z_s^{1,u}, u_s) ds + G(s, X_s^{t,x,u}, Y_s^{1,u}) dK_s^{t,x,u} \\ &\quad - Z_s^{1,u} dB_s, \quad s \in [t, t + \alpha]; \\ Y_{t+\alpha}^{1,u} &= 0, \end{cases} \quad (4.31)$$

where the pair of processes $(X^{t,x,u}, K^{t,x,u})$ are given by (3.1) and $u(\cdot) \in \mathcal{U}_{t,t+\alpha}$. Therefore, Lemma 4.1 still holds for $x \in \bar{D}$.

On the other hand, from the DPP (see Theorem 3.1), for every α ,

$$\varphi(t, x) = W(t, x) = \operatorname{esssup}_{u \in \mathcal{U}_{t,t+\alpha}} G_{t,t+\alpha}^{t,x,u}[W(t + \alpha, X_{t+\alpha}^{t,x,u})],$$

and from $W \geq \varphi$ and the monotonicity property of $G_{t,t+\delta}^{t,x;u}[\cdot]$ (see Lemma 2.7) we have

$$\operatorname{esssup}_{u \in \mathcal{U}_{t,t+\alpha}} \{G_{t,t+\delta}^{t,x;u}[\varphi(t+\alpha, X_{t+\alpha}^{t,x;u})] - \varphi(t, x)\} \leq 0, \text{ P-a.s.}$$

Thus, from Lemma 4.1, we get

$$\operatorname{esssup}_{u \in \mathcal{U}_{t,t+\alpha}} Y_t^{1,u} \leq 0, \text{ P-a.s.}$$

Hence, P-a.s.,

$$Y_t^{1,u^*} \leq 0. \quad (4.32)$$

Remark 4.3. Similarly, for $x \in \partial D$ we also have (4.32).

For $u^* \in \mathcal{U}_{t,t+\alpha}$ we define

$$\tau = \inf \{s \geq t : |X_s^{t,x;u^*} - x| \geq \bar{\alpha}\} \wedge (t + \alpha).$$

Consequently, on $[t, \tau]$ the process $(K^{t,x;u^*})$ is zero and, hence

$$Y_s^{1;u^*} = Y_\tau^{1;u^*} + \int_s^\tau F(r, X_r^{t,x;u^*}, Y_r^{1;u^*}, Z_r^{1;u^*}, u^*) dr - \int_s^\tau Z_r^{1;u^*} dB_r.$$

Consider the following two BSDEs:

$$\begin{cases} -d\hat{Y}_s^2 &= [-C^*(|\hat{Y}_s^2| + |\hat{Z}_s^2|) + \frac{1}{2}\theta] ds - \hat{Z}_s^2 dB_s, \\ \hat{Y}_{t+\alpha}^2 &= 0 \end{cases} \quad (4.33)$$

whose unique solution is given by

$$\hat{Y}_s^2 = \frac{\theta}{2C^*} \left(1 - e^{C^*(s-(t+\alpha))}\right), \quad \hat{Z}_s^2 = 0, \quad s \in [t, t+\alpha], \quad (4.34)$$

and

$$\begin{cases} -d\hat{Y}_s^3 &= [-C^*(|\hat{Y}_s^3| + |\hat{Z}_s^3|) + \frac{1}{2}\theta] ds - \hat{Z}_s^3 dB_s, \quad s \in [t, \tau]; \\ \hat{Y}_\tau^3 &= Y_\tau^{1;u^*}. \end{cases} \quad (4.35)$$

We have the following lemma.

Lemma 4.3. We have $Y_t^{1,u^*} \geq \hat{Y}_t^3$ and $|\hat{Y}_t^2 - \hat{Y}_t^3| \leq C\alpha^{\frac{3}{2}}$, P-a.s. Here $C > 0$ is independent of both the control u and α .

Proof. (1) We observe from (4.30) and the definition of τ that, for all $(s, y, z, u) \in [t, \tau] \times \mathbb{R} \times \mathbb{R}^d \times U$,

$$\begin{aligned} F(s, X_s^{t,x;u^*}, y, z, u^*) &\geq -C^*(|y| + |z|) + F(s, X_s^{t,x;u^*}, 0, 0, u^*) \\ &\geq -C^*(|y| + |z|) + \frac{1}{2}\theta. \end{aligned}$$

Consequently, from Lemma 2.2 (the comparison result for BSDEs) we have that

$$Y_s^{1,u^*} \geq \hat{Y}_s^3, \quad s \in [t, \tau],$$

where \widehat{Y}^3 is defined by BSDE (4.35).

(2) From the equation (4.31), Propositions 5.1 and 5.2

$$\left| Y_{\tau}^{1;u^*} \right| \leq C(t + \alpha - \tau)^{\frac{1}{2}} + C \left(E \left[(K_{t+\alpha}^{t,x;u^*} - K_{\tau}^{t,x;u^*})^2 \mid \mathcal{F}_{\tau} \right] \right)^{\frac{1}{2}},$$

where C is independent of controls. Then similar to the proof of estimate (4.14), we have

$$E \left[\left| Y_{\tau}^{1;u^*} \right|^2 \mid \mathcal{F}_t \right] \leq C \left(E \left[(t + \alpha - \tau)^2 \mid \mathcal{F}_t \right] \right)^{\frac{1}{2}}. \quad (4.36)$$

Similar to (4.16), we still have

$$P \{ \tau < t + \alpha \mid \mathcal{F}_t \} \leq \frac{C}{\bar{\alpha}^8} \alpha^4. \quad (4.37)$$

Therefore,

$$E \left[\left| Y_{\tau}^{1;u^*} \right|^2 \mid \mathcal{F}_t \right] \leq C \alpha \left(P \{ \tau < t + \alpha \mid \mathcal{F}_t \} \right)^{\frac{1}{2}} \leq \frac{C}{\bar{\alpha}^4} \alpha^3. \quad (4.38)$$

Furthermore, from Lemma 2.3

$$\begin{aligned} \left| \widehat{Y}_t^2 - \widehat{Y}_t^3 \right| &\leq C \left(E[|\widehat{Y}_{\tau}^2 - \widehat{Y}_{\tau}^3|^2 \mid \mathcal{F}_t] \right)^{\frac{1}{2}} \\ &\leq C \left(E[|\widehat{Y}_{\tau}^2|^2 \mid \mathcal{F}_t] \right)^{\frac{1}{2}} + C \left(E[|\widehat{Y}_{\tau}^3|^2 \mid \mathcal{F}_t] \right)^{\frac{1}{2}} \\ &\leq C \frac{\theta}{2} (1 - e^{-C^* \alpha}) (P \{ \tau < t + \alpha \mid \mathcal{F}_t \})^{\frac{1}{2}} + C \left(E[|Y_{\tau}^{1;u^*}|^2 \mid \mathcal{F}_t] \right)^{\frac{1}{2}} \\ &\leq C \frac{\theta}{2} (1 - e^{-C^* \alpha}) \frac{1}{\bar{\alpha}^4} \alpha^2 + \frac{C}{\bar{\alpha}^2} \alpha^{\frac{3}{2}} \\ &\leq C \alpha^{\frac{3}{2}}, \end{aligned} \quad (4.39)$$

for any $\alpha \in (0, \bar{\alpha}]$.

Now we complete the proof of Proposition 4.2.

Proof of Proposition 4.2 (sequel).

By combining (4.32) with Lemma 4.3 we then obtain

$$0 \geq Y_t^{1,u^*} \geq \widehat{Y}_t^3 \geq \widehat{Y}_t^2 - |\widehat{Y}_t^2 - \widehat{Y}_t^3| \geq \widehat{Y}_t^2 - C \alpha^{\frac{3}{2}}, \text{P-a.s.}$$

i.e.,

$$0 \geq Y_t^{1,u^*} \geq \frac{\theta}{2C^*} \left(1 - e^{-C^* \alpha} \right) - C \alpha^{\frac{3}{2}}, \text{P-a.s.}$$

Therefore,

$$0 \geq \frac{\theta}{2C^*} \frac{1 - e^{-C^* \alpha}}{\alpha} - C \alpha^{\frac{1}{2}}.$$

Letting $\alpha \rightarrow 0+$, we get $0 \geq \frac{\theta}{2}$, which contradicts our assumption that $\theta > 0$. Therefore, we have

$$\sup_{u \in U} F(t, x, 0, 0, u) \leq 0,$$

which implies by the definition of F that

$$\frac{\partial \varphi}{\partial t}(t, x) + H(t, x, W, D\varphi, D^2\varphi) \leq 0, \quad \text{if } x \in D.$$

We now consider the case $x \in \partial D$. We must prove that

$$\min \left\{ \frac{\partial \varphi}{\partial t}(t, x) + H(t, x, \varphi, D\varphi, D^2\varphi), \quad \frac{\partial \varphi}{\partial n}(t, x) + g(t, x, \varphi) \right\} \leq 0$$

If this is not true, then there exists some $\theta > 0$ such that

$$\sup_{u \in U} F(t, x, 0, 0, u) \geq \theta > 0, \quad G(t, x, 0) \geq \theta > 0, \quad (4.40)$$

therefore, there exists $u^* \in U$ such that

$$F(t, x, 0, 0, u^*) \geq \frac{2\theta}{3}.$$

Choose $\bar{\alpha} \in (0, T - t]$ such that

$$F(s, y, 0, 0, u^*) \geq \frac{1}{2}\theta, \quad (4.41)$$

$$G(s, y, 0) \geq \frac{1}{2}\theta, \quad \text{for all } t \leq s \leq t + \bar{\alpha}, \quad |y - x| \leq \bar{\alpha}. \quad (4.42)$$

Now we fix $\bar{\alpha}$, and we consider any $\alpha \in (0, \bar{\alpha}]$. Similarly, we still consider GBSDE (4.31) with $x \in \partial D$. For this $u^* \in \mathcal{U}_{t, t+\alpha}$ we still have (4.32) and define

$$\tau = \inf \{s \geq t : |X_s^{t, x; u^*} - x| \geq \bar{\alpha}\} \wedge (t + \alpha);$$

We observe that, for all $(s, y, z) \in [t, \tau] \times \mathbb{R} \times \mathbb{R}^d$, from (4.41), (4.42) and the definition of τ

$$\begin{aligned} F(s, X_s^{t, x; u^*}, y, z, u_s^\varepsilon) &\geq -C^*(|y| + |z|) + F(s, X_s^{t, x; u^*}, 0, 0, u_s^\varepsilon) \\ &\geq -C^*(|y| + |z|) + \frac{1}{2}\theta. \\ G(s, X_s^{t, x; u^*}, y) &\geq -C^*|y| + G(s, X_s^{t, x; u^*}, 0) \\ &\geq -C^*|y| + \frac{1}{2}\theta. \end{aligned}$$

Consequently, from the comparison result for GBSDEs (Lemma 2.7, or Remark 1.5 in Pardoux and Zhang [17]) we have that

$$Y_s^{1, u^*} \geq \widehat{Y}_s^4, \quad s \in [t, \tau], \quad \text{P-a.s.,}$$

where \widehat{Y}^4 is defined by the following BSDE:

$$\begin{cases} -d\widehat{Y}_s^4 = \{-C^*(|\widehat{Y}_s^4| + |\widehat{Z}_s^4|) + \frac{1}{2}\theta\}ds + (-C^*|\widehat{Y}_s^4| + \frac{1}{2}\theta)dK_s^{t, x; u^*} - \widehat{Z}_s^4 dB_s, \\ \widehat{Y}_\tau^4 = Y_\tau^{1, u^*}. \end{cases} \quad (4.43)$$

On the other hand, we also have to introduce the following BSDE:

$$\begin{cases} -d\widehat{Y}_s^5 = \{-C^*(|\widehat{Y}_s^5| + |\widehat{Z}_s^5|) + \frac{1}{2}\theta\}ds + (-C^*|\widehat{Y}_s^5| + \frac{1}{2}\theta)dK_s^{t, x; u^*} - \widehat{Z}_s^5 dB_s, \\ \widehat{Y}_{t+\alpha}^5 = 0. \end{cases} \quad (4.44)$$

Notice that $-C^*|\widehat{Y}_s^2| + \frac{1}{2}\theta > 0$, therefore $\widehat{Y}_s^5 \geq \widehat{Y}_s^2$, $s \in [t, t + \alpha]$, P-a.s., from Lemma 2.7.

From Lemma 2.6 we have

$$\begin{aligned}
|\widehat{Y}_t^4 - \widehat{Y}_t^5| &\leq C \left(E[|\widehat{Y}_\tau^4 - \widehat{Y}_\tau^5|^2 | \mathcal{F}_t] \right)^{\frac{1}{2}} \\
&\leq C \left(E[|\widehat{Y}_\tau^4|^2 | \mathcal{F}_t] \right)^{\frac{1}{2}} + C \left(E[|\widehat{Y}_\tau^5|^2 | \mathcal{F}_t] \right)^{\frac{1}{2}} \\
&\leq C \left(E[|Y_\tau^{1;u^*}|^2 | \mathcal{F}_t] \right)^{\frac{1}{2}} + C \left(E[|\widehat{Y}_\tau^2|^2 | \mathcal{F}_t] \right)^{\frac{1}{2}} + C \left(E[|\widehat{Y}_\tau^5 - \widehat{Y}_\tau^2|^2 | \mathcal{F}_t] \right)^{\frac{1}{2}} \\
&\leq C\alpha^{\frac{3}{2}} + C \left(E[|\widehat{Y}_\tau^5 - \widehat{Y}_\tau^2|^2 | \mathcal{F}_t] \right)^{\frac{1}{2}} \quad (\text{from the proof of (4.39)}),
\end{aligned} \tag{4.45}$$

for any $\alpha \in (0, \bar{\alpha}]$.

Similar to (4.25) and (4.26), $P\{\tau < t + \alpha | \mathcal{F}_t\} \leq \frac{C}{\bar{\alpha}^8} \alpha^4$; and

$$\begin{aligned}
&E[|\widehat{Y}_\tau^5 - \widehat{Y}_\tau^2|^2 | \mathcal{F}_t] \\
&\leq CE \left[\int_\tau^{t+\alpha} e^{2K_s^{t,x;u^*}} (C^*|\widehat{Y}_s^2| - \frac{1}{2}\theta)^2 dK_s^{t,x;u^*} | \mathcal{F}_t \right] \\
&= CE \left[\int_\tau^{t+\alpha} e^{2K_s^{t,x;u^*}} \frac{\theta^2}{4} e^{2C^*(s-(t+\alpha))} dK_s^{t,x;u^*} | \mathcal{F}_t \right] \\
&\leq C\theta^2 \alpha^{\frac{5}{2}}.
\end{aligned} \tag{4.46}$$

Therefore,

$$|\widehat{Y}_t^4 - \widehat{Y}_t^5| \leq C\alpha^{\frac{3}{2}} + C\theta\alpha^{\frac{5}{4}}. \tag{4.47}$$

Now we obtain

$$0 \geq Y_t^{1,u^*} \geq \widehat{Y}_t^4 \geq \widehat{Y}_t^5 - |\widehat{Y}_t^4 - \widehat{Y}_t^5| \geq \widehat{Y}_t^2 - C\alpha^{\frac{3}{2}} - C\theta\alpha^{\frac{5}{4}}, \text{P-a.s.}$$

i.e.,

$$0 \geq Y_t^{1,u^*} \geq \frac{\theta}{2C^*} \left(1 - e^{-C^*\alpha} \right) - C\alpha^{\frac{3}{2}} - C\theta\alpha^{\frac{5}{4}}, \text{P-a.s.}$$

Therefore,

$$0 \geq \frac{\theta}{2C^*} \frac{1 - e^{-C^*\alpha}}{\alpha} - C\alpha^{\frac{1}{2}} - C\theta\alpha^{\frac{1}{4}},$$

and by taking the limit as $\alpha \downarrow 0$, we get $0 \geq \frac{\theta}{2}$ which contradicts our assumption that $\theta > 0$. Therefore, it must hold

$$\min \left\{ \frac{\partial \varphi}{\partial t}(t, x) + H(t, x, W, D\varphi, D^2\varphi), \quad \frac{\partial \varphi}{\partial n}(t, x) + g(t, x, W) \right\} \leq 0.$$

□

Therefore, we have

Theorem 4.1. *Under the assumptions (H4.1) and (H4.2), the value function W is a viscosity solution to (4.1).*

Remark 4.4. *Uniqueness of viscosity solutions to elliptic equations with nonlinear Neumann boundary condition can be found in Barles [1], Bourgoing [3], and Crandall, Ishii, and Lions [5, Section 7B], etc. Uniqueness of viscosity solutions to a system of parabolic PDEs in the whole space can be found in Buckdahn and Li [4].*

5 Appendix: Forward-Backward SDES (FBSDEs)

In this section we document some necessary basic results on GBSDEs associated with forward reflected SDEs (for short: FSDEs).

We consider measurable functions $b : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^d$ and $\sigma : [0, T] \times \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}^{d \times d}$ which are supposed to satisfy the following conditions:

- (H5.1) (i) $b(\cdot, 0)$ and $\sigma(\cdot, 0)$ are \mathbb{F} -adapted processes, and there exists some constant $C > 0$ such that
 $|b(t, x)| + |\sigma(t, x)| \leq C(1 + |x|)$, a.s., for all $0 \leq t \leq T$, $x \in \mathbb{R}^d$;
(ii) b and σ are Lipschitz in x , i.e., there is some constant $C > 0$ such that
 $|b(t, x) - b(t, x')| + |\sigma(t, x) - \sigma(t, x')| \leq C|x - x'|$, a.s.,
for all $0 \leq t \leq T$, $x, x' \in \mathbb{R}^d$.

Under the assumption (H5.1), it follows from the results in Lions and Sznitman [12] that for each initial condition $(t, \zeta) \in [0, T] \times L^2(\Omega, \mathcal{F}_t, P; \bar{D})$ there exists a unique pair of progressively measurable continuous processes $\{(X^{t, \zeta}, K^{t, \zeta})\}$, with values in $\bar{D} \times \mathbb{R}_+$, such that

$$\begin{cases} X_s^{t, \zeta} &= \zeta + \int_t^s b(r, X_r^{t, \zeta}) dr + \int_t^s \sigma(r, X_r^{t, \zeta}) dB_r + \int_t^s \nabla \phi(X_r^{t, \zeta}) dK_r^{t, \zeta}, \quad s \in [t, T], \\ K_s^{t, \zeta} &= \int_t^s I_{\{X_r^{t, \zeta} \in \partial D\}} dK_r^{t, \zeta}, \quad K^{t, \zeta} \text{ is increasing.} \end{cases} \quad (5.1)$$

Proposition 5.1. *For each $T \geq 0$, there exists a constant C_T such that for all $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \bar{D})$,*

$$E\left(\sup_{t \leq s \leq T} |X_s^{t, \zeta} - X_s^{t, \zeta'}|^4 \middle| \mathcal{F}_t\right) \leq C_T |\zeta - \zeta'|^4, \quad (5.2)$$

and

$$E\left(\sup_{t \leq s \leq T} |K_s^{t, \zeta} - K_s^{t, \zeta'}|^4 \middle| \mathcal{F}_t\right) \leq C_T |\zeta - \zeta'|^4. \quad (5.3)$$

Moreover, for each $\mu > 0, s \in [t, T]$, there exists $C(\mu, s)$ such that for all $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \bar{D})$,

$$E(e^{\mu K_s^{t, \zeta}} | \mathcal{F}_t) \leq C(\mu, s). \quad (5.4)$$

The proof is similar to that of Propositions 3.1 and 3.2 in Pardoux and Zhang [17].

Assume that the three functions $f(t, x, y, z), g(t, x, y)$ and $\Phi(x)$ satisfy the following conditions:

- (H5.2) (i) $\Phi : \Omega \times \mathbb{R}^d \rightarrow \mathbb{R}$ is an $\mathcal{F}_T \otimes \mathcal{B}(\mathbb{R}^d)$ -measurable random variable and
 $f : [0, T] \times \Omega \times \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d \rightarrow \mathbb{R}$ is a measurable process such that
 $f(\cdot, x, y, z)$ is \mathbb{F} -adapted, for all $(x, y, z) \in \mathbb{R}^d \times \mathbb{R} \times \mathbb{R}^d$;
 $g : [0, T] \times \mathbb{R}^d \times \mathbb{R} \rightarrow \mathbb{R}$ is a measurable function such that
 $g(\cdot) \in C^{1,2,2}([0, T] \times \mathbb{R}^d \times \mathbb{R})$;
(ii) There exists a constant $C > 0$ such that
 $|f(t, x, y, z) - f(t, x', y', z')| + |g(t, x, y) - g(t, x', y')| + |\Phi(x) - \Phi(x')|$
 $\leq C(|x - x'| + |y - y'| + |z - z'|)$, a.s.,
for all $0 \leq t \leq T$; $x, x' \in \mathbb{R}^d$; $y, y' \in \mathbb{R}$; $z, z' \in \mathbb{R}^d$;

- (iii) f and Φ satisfy a linear growth condition, i.e., there exists some $C > 0$ such that, $\text{dt} \times \text{dP}$ -a.e., for all $x \in \mathbb{R}^d$, $|f(t, x, 0, 0)| + |\Phi(x)| \leq C(1 + |x|)$.

Under the above assumptions the coefficients $f(s, X_s^{t,\zeta}, y, z)$ and $g(s, X_s^{t,\zeta}, y)$ satisfy (H2.1) and $\xi = \Phi(X_T^{t,\zeta}) \in L^2(\Omega, \mathcal{F}_T, P)$. Therefore, the following GBSDE possesses a unique solution:

$$\begin{cases} -dY_s^{t,\zeta} &= f(s, X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta})ds + g(s, X_s^{t,\zeta}, Y_s^{t,\zeta})dK_s^{t,\zeta} - Z_s^{t,\zeta}dB_s, \quad s \in [t, T], \\ Y_T^{t,\zeta} &= \Phi(X_T^{t,\zeta}). \end{cases} \quad (5.5)$$

Proposition 5.2. *Let assumptions (H5.1) and (H5.2) hold. Then, for any $0 \leq t \leq T$ and $\zeta, \zeta' \in L^2(\Omega, \mathcal{F}_t, P; \bar{D})$,*

- (i) $E[\sup_{t \leq s \leq T} |Y_s^{t,\zeta}|^2 + \int_t^T |Z_s^{t,\zeta}|^2 ds | \mathcal{F}_t] \leq C(1 + |\zeta|^2), \text{ a.s.};$ and in particular, $|Y_t^{t,\zeta}| \leq C(1 + |\zeta|), \text{ a.s.};$
- (ii) $|Y_t^{t,\zeta} - Y_t^{t,\zeta'}| \leq C|\zeta - \zeta'| + C|\zeta - \zeta'|^{\frac{1}{2}}, \text{ a.s.},$

where the constant $C > 0$ depends only on the Lipschitz and the growth constants of b, σ, f, g and Φ .

Remark 5.1. *Since D is bounded, we have*

$$E[\sup_{t \leq s \leq T} |Y_s^{t,\zeta}|^2 + \int_t^T |Z_s^{t,\zeta}|^2 ds | \mathcal{F}_t] \leq C, \text{ a.s.}, \quad (5.6)$$

where C is independent of ζ .

Proof. From Lemma 2.5 and Proposition 5.1, we have assertion (i). Now we prove assertion (ii). First notice that from (i) we have $|Y_t^{t,\zeta}| \leq C(1 + |\zeta|), \text{ a.s.}$, therefore we can get from the uniqueness of the solution of equations (5.1) and (5.5) that

$$|Y_s^{t,\zeta}| = |Y_s^{s, X_s^{t,\zeta}}| \leq C(1 + |X_s^{t,\zeta}|) \leq C, \text{ a.s.}, \quad (5.7)$$

since D is bounded. From Burkholder-Davis-Gundy inequality and (5.5), as well as from the boundedness of the processes $X^{t,\zeta}, Y^{t,\zeta}$,

$$\begin{aligned} & E[(\int_s^T |Z_r^{t,\zeta}|^2 dr)^2 | \mathcal{F}_t] \\ & \leq CE[\sup_{r \in [s, T]} |\int_s^r Z_v^{t,\zeta} dB_v|^4 | \mathcal{F}_t] \\ & \leq C + C_0(T - s)^2 E[(\int_s^T |Z_r^{t,\zeta}|^2 dr)^2 | \mathcal{F}_t] + CE[(K_T^{t,\zeta})^4 | \mathcal{F}_t] \\ & \leq C + C_0(T - s)^2 E[(\int_s^T |Z_r^{t,\zeta}|^2 dr)^2 | \mathcal{F}_t]. \end{aligned}$$

Consequently, for $T - s \leq (\frac{1}{2C_0})^{1/2}$, $E[(\int_s^T |Z_r^{t,\zeta}|^2 dr)^2 | \mathcal{F}_t] \leq C$. This argument allows to choose a partition $t = t_0 < t_1 < \dots < t_N = T$ of the interval $[t, T]$ such that $E[(\int_{t_{i-1}}^{t_i} |Z_r^{t,\zeta}|^2 dr)^2 | \mathcal{F}_t] \leq C, 1 \leq i \leq N$. Therefore, we have

$$E[(\int_t^T |Z_r^{t,\zeta}|^2 dr)^2 | \mathcal{F}_t] \leq C. \quad (5.8)$$

For any $\lambda > 0$, applying Itô's formula to $e^{\lambda K_s^{t,\zeta'}} |Y_s^{t,\zeta} - Y_s^{t,\zeta'}|^2$, we have

$$\begin{aligned}
& |Y_s^{t,\zeta} - Y_s^{t,\zeta'}|^2 + \lambda \int_s^T e^{\lambda K_r^{t,\zeta'}} |Y_r^{t,\zeta} - Y_r^{t,\zeta'}|^2 dK_r^{t,\zeta'} + \int_s^T e^{\lambda K_r^{t,\zeta'}} |Z_r^{t,\zeta} - Z_r^{t,\zeta'}|^2 dr \\
= & e^{\lambda K_T^{t,\zeta'}} |Y_T^{t,\zeta} - Y_T^{t,\zeta'}|^2 \\
& + 2 \int_s^T e^{\lambda K_r^{t,\zeta'}} (Y_r^{t,\zeta} - Y_r^{t,\zeta'}) (f(r, X_r^{t,\zeta}, Y_r^{t,\zeta}, Z_r^{t,\zeta}) - f(r, X_r^{t,\zeta'}, Y_r^{t,\zeta'}, Z_r^{t,\zeta'})) dr \\
& + 2 \int_s^T e^{\lambda K_r^{t,\zeta'}} (Y_r^{t,\zeta} - Y_r^{t,\zeta'}) (g(r, X_r^{t,\zeta}, Y_r^{t,\zeta}) - g(r, X_r^{t,\zeta'}, Y_r^{t,\zeta'})) dK_r^{t,\zeta'} \\
& + 2 \int_s^T e^{\lambda K_r^{t,\zeta'}} (Y_r^{t,\zeta} - Y_r^{t,\zeta'}) g(r, X_r^{t,\zeta}, Y_r^{t,\zeta}) d(K_r^{t,\zeta} - K_r^{t,\zeta'}) \\
& - 2 \int_s^T e^{\lambda K_r^{t,\zeta'}} (Y_r^{t,\zeta} - Y_r^{t,\zeta'}) < Z_r^{t,\zeta} - Z_r^{t,\zeta'}, dB_r >.
\end{aligned} \tag{5.9}$$

Then from (H5.1), (H5.2), (5.4), (5.7) and (5.8), taking a suitable $\lambda > 0$, we get:

$$\begin{aligned}
& |Y_s^{t,\zeta} - Y_s^{t,\zeta'}|^2 \\
\leq & C|\zeta - \zeta'|^2 + CE[\int_s^T e^{\lambda K_r^{t,\zeta'}} |Y_r^{t,\zeta} - Y_r^{t,\zeta'}|^2 dr | \mathcal{F}_s] \\
& + CE[\int_s^T e^{\lambda K_r^{t,\zeta'}} |X_r^{t,\zeta} - X_r^{t,\zeta'}|^2 dr | \mathcal{F}_s] + CE[\int_s^T e^{\lambda K_r^{t,\zeta'}} |X_r^{t,\zeta} - X_r^{t,\zeta'}|^2 dK_r^{t,\zeta'} | \mathcal{F}_s] \\
& + E[2 \int_s^T e^{\lambda K_r^{t,\zeta'}} (Y_r^{t,\zeta} - Y_r^{t,\zeta'}) g(r, X_r^{t,\zeta}, Y_r^{t,\zeta}) d(K_r^{t,\zeta} - K_r^{t,\zeta'}) | \mathcal{F}_s].
\end{aligned} \tag{5.10}$$

Furthermore, from Proposition 5.1, we have

$$\begin{aligned}
& |Y_s^{t,\zeta} - Y_s^{t,\zeta'}|^2 \\
\leq & C|\zeta - \zeta'|^2 + CE[\int_s^T e^{\lambda K_r^{t,\zeta'}} |Y_r^{t,\zeta} - Y_r^{t,\zeta'}|^2 dr | \mathcal{F}_s] \\
& + E[2 \int_s^T e^{\lambda K_r^{t,\zeta'}} (Y_r^{t,\zeta} - Y_r^{t,\zeta'}) g(r, X_r^{t,\zeta}, Y_r^{t,\zeta}) d(K_r^{t,\zeta} - K_r^{t,\zeta'}) | \mathcal{F}_s].
\end{aligned} \tag{5.11}$$

On the other hand, applying Itô's formula to $e^{\lambda K_s^{t,\zeta'}} (Y_s^{t,\zeta} - Y_s^{t,\zeta'}) g(s, X_s^{t,\zeta}, Y_s^{t,\zeta}) (K_s^{t,\zeta} - K_s^{t,\zeta'})$, we have

$$\begin{aligned}
& E[\int_s^T e^{\lambda K_r^{t,\zeta'}} (Y_r^{t,\zeta} - Y_r^{t,\zeta'}) g(r, X_r^{t,\zeta}, Y_r^{t,\zeta}) d(K_r^{t,\zeta} - K_r^{t,\zeta'}) | \mathcal{F}_s] \\
= & E[e^{\lambda K_T^{t,\zeta'}} (Y_T^{t,\zeta} - Y_T^{t,\zeta'}) g(T, X_T^{t,\zeta}, Y_T^{t,\zeta}) (K_T^{t,\zeta} - K_T^{t,\zeta'}) | \mathcal{F}_s] \\
& + E[\int_s^T f_1(r) (K_r^{t,\zeta'} - K_r^{t,\zeta}) dr | \mathcal{F}_s] \\
& + E[\int_s^T f_2(r) (K_r^{t,\zeta'} - K_r^{t,\zeta}) dK_r^{t,\zeta'} | \mathcal{F}_s] \\
& + E[\int_s^T f_3(r) (K_r^{t,\zeta'} - K_r^{t,\zeta}) dK_r^{t,\zeta} | \mathcal{F}_s],
\end{aligned} \tag{5.12}$$

where

$$\begin{aligned}
f_1(s) = & -e^{\lambda K_s^{t,\zeta'}} (f(s, X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta}) - f(s, X_s^{t,\zeta'}, Y_s^{t,\zeta'}, Z_s^{t,\zeta'})) g(s, X_s^{t,\zeta}, Y_s^{t,\zeta}) \\
& + e^{\lambda K_s^{t,\zeta'}} (Y_s^{t,\zeta} - Y_s^{t,\zeta'}) \{ \frac{\partial}{\partial s} g(s, X_s^{t,\zeta}, Y_s^{t,\zeta}) + \nabla_x g(s, X_s^{t,\zeta}, Y_s^{t,\zeta}) b(s, X_s^{t,\zeta}) \\
& - \nabla_y g(s, X_s^{t,\zeta}, Y_s^{t,\zeta}) f(s, X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta}) + \frac{1}{2} \text{tr}(D_x^2 g(s, X_s^{t,\zeta}, Y_s^{t,\zeta}) \sigma \sigma^T(s, X_s^{t,\zeta})) \\
& + \frac{1}{2} D_y^2 g(s, X_s^{t,\zeta}, Y_s^{t,\zeta}) |Z_s^{t,\zeta}|^2 + \frac{1}{2} \text{tr} < D_{xy} g(s, X_s^{t,\zeta}, Y_s^{t,\zeta}) \sigma(s, X_s^{t,\zeta}), Z_s^{t,\zeta} > \} \\
& + e^{\lambda K_s^{t,\zeta'}} (Z_s^{t,\zeta} - Z_s^{t,\zeta'}) \{ \nabla_x g(s, X_s^{t,\zeta}, Y_s^{t,\zeta}) \sigma(s, X_s^{t,\zeta}) + \nabla_y g(s, X_s^{t,\zeta}, Y_s^{t,\zeta}) Z_s^{t,\zeta} \};
\end{aligned}$$

$$f_2(s) = \{ \lambda e^{\lambda K_s^{t,\zeta'}} (Y_s^{t,\zeta} - Y_s^{t,\zeta'}) + e^{\lambda K_s^{t,\zeta'}} g(s, X_s^{t,\zeta'}, Y_s^{t,\zeta'}) \} g(s, X_s^{t,\zeta}, Y_s^{t,\zeta});$$

$$\begin{aligned}
f_3(s) = & e^{\lambda K_s^{t,\zeta'}} (Y_s^{t,\zeta} - Y_s^{t,\zeta'}) \{ \nabla_x g(s, X_s^{t,\zeta}, Y_s^{t,\zeta}) \nabla \phi(s, X_s^{t,\zeta}) - \nabla_y g(s, X_s^{t,\zeta}, Y_s^{t,\zeta}) g(s, X_s^{t,\zeta}, Y_s^{t,\zeta}) \} \\
& - e^{\lambda K_s^{t,\zeta'}} |g(s, X_s^{t,\zeta}, Y_s^{t,\zeta})|^2.
\end{aligned}$$

From assertion (i), Propositions 5.1, (5.7) and (5.8), we have

$$E\left[\int_s^T e^{\lambda K_r^{t,\zeta'}} (Y_r^{t,\zeta} - Y_r^{t,\zeta'}) g(r, X_r^{t,\zeta}, Y_r^{t,\zeta}) d(K_r^{t,\zeta} - K_r^{t,\zeta'}) | \mathcal{F}_s\right] \leq C|\zeta - \zeta'|^2 + C|\zeta - \zeta'|.$$

Furthermore, from (5.11) and (5.4) we have

$$\begin{aligned} & E[|Y_s^{t,\zeta} - Y_s^{t,\zeta'}|^4 | \mathcal{F}_t] \\ & \leq C|\zeta - \zeta'|^4 + C|\zeta - \zeta'|^2 + CE[(\int_s^T e^{\lambda K_r^{t,\zeta'}} |Y_r^{t,\zeta} - Y_r^{t,\zeta'}|^2 dr)^2 | \mathcal{F}_t] \\ & \leq C|\zeta - \zeta'|^4 + C|\zeta - \zeta'|^2 + CE[e^{2\lambda K_T^{t,\zeta'}} | \mathcal{F}_t] E[\int_s^T |Y_r^{t,\zeta} - Y_r^{t,\zeta'}|^4 dr | \mathcal{F}_t] \\ & \leq C|\zeta - \zeta'|^4 + C|\zeta - \zeta'|^2 + CE[\int_s^T |Y_r^{t,\zeta} - Y_r^{t,\zeta'}|^4 dr | \mathcal{F}_t], \quad s \in [t, T], \end{aligned}$$

then from Gronwall's Lemma, we get $E[|Y_s^{t,\zeta} - Y_s^{t,\zeta'}|^4 | \mathcal{F}_t] \leq C|\zeta - \zeta'|^4 + C|\zeta - \zeta'|^2$, a.s, $s \in [t, T]$, which means (ii) for $s = t$. \square

Remark 5.2. If g is a bounded random variable, assertion (ii) of (5.6) still holds. Indeed, from Lemma 2.3 and Proposition 5.1, we get

$$\begin{aligned} |Y_t^{t,\zeta} - Y_t^{t,\zeta'}|^2 & \leq CE[|\zeta - \zeta' + g(\omega)(K_T^{t,\zeta} - K_T^{t,\zeta'})|^2 | \mathcal{F}_t] \\ & \quad + CE[\int_t^T |f(s, X_s^{t,\zeta}, Y_s^{t,\zeta}, Z_s^{t,\zeta}) - f(s, X_s^{t,\zeta'}, Y_s^{t,\zeta'}, Z_s^{t,\zeta'})|^2 ds | \mathcal{F}_t] \\ & \leq C|\zeta - \zeta'|^2, \quad a.s. \end{aligned}$$

Proposition 5.3. Let Hypotheses (H5.1) and (H5.2) hold. Then, for any $0 \leq \alpha \leq T - t$ and the associated initial conditions $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \bar{D})$, we have the following estimates:

$$E \left[|K_{t+\alpha}^{t,\zeta}|^2 | \mathcal{F}_t \right] \leq C\alpha, \quad a.s., \quad (5.13)$$

where the constant $C > 0$ depends only on the Lipschitz and the growth constants of b , σ , f , g and Φ .

Proof. For $\zeta' \in L^2(\Omega, \mathcal{F}_t, P; \bar{D})$, from Itô's formula we have

$$\begin{aligned} |X_s^{t,\zeta} - \zeta'|^2 & = |\zeta - \zeta'|^2 + 2 \int_t^s (X_r^{t,\zeta} - \zeta') b(r, X_r^{t,\zeta}) dr + 2 \int_t^s (X_r^{t,\zeta} - \zeta') \sigma(r, X_r^{t,\zeta}) dB_r \\ & \quad + \int_t^s |\sigma(r, X_r^{t,\zeta})|^2 dr + 2 \int_t^s (X_r^{t,\zeta} - \zeta') \nabla \phi(X_r^{t,\zeta}) dK_r^{t,\zeta}, \quad s \in [t, T]. \end{aligned} \quad (5.14)$$

Since $D \subset \mathbb{R}^d$ is convex, we have

$$\int_t^s (X_r^{t,\zeta} - \zeta') \nabla \phi(X_r^{t,\zeta}) dK_r^{t,\zeta} \leq 0. \quad (5.15)$$

Therefore, we have

$$E\left[\sup_{s \in [t, t+\alpha]} |X_s^{t,\zeta} - \zeta'|^2 | \mathcal{F}_t\right] \leq C(|\zeta - \zeta'|^2 + \alpha).$$

Recall that D is an open connected bounded convex subset. In particular, we have,

$$E\left[\sup_{s \in [t, t+\alpha]} |X_s^{t,\zeta} - \zeta|^2 | \mathcal{F}_t\right] \leq C\alpha. \quad (5.16)$$

Because $\phi \in C_b^2(\mathbb{R}^d)$ we have

$$\begin{aligned}\phi(X_s^{t,\zeta}) &= \phi(\zeta) + \int_t^s \nabla \phi(X_r^{t,\zeta}) b(r, X_r^{t,\zeta}) dr + \int_t^s \nabla \phi(X_r^{t,\zeta}) \sigma(r, X_r^{t,\zeta}) dB_r \\ &\quad + \frac{1}{2} \int_t^s \text{tr}(D^2 \phi \sigma(r, X_r^{t,\zeta}) \sigma^T(r, X_r^{t,\zeta})) dr + \int_t^s |\nabla \phi(X_r^{t,\zeta})|^2 dK_r^{t,\zeta}, \quad s \in [t, T].\end{aligned}$$

Therefore, we get

$$K_s^{t,\zeta} \leq |\phi(X_s^{t,\zeta}) - \phi(\zeta)| + C \int_t^s (1 + |X_r^{t,\zeta}|^2) dr + \left| \int_t^s \nabla \phi(X_r^{t,\zeta}) \sigma(r, X_r^{t,\zeta}) dB_r \right|,$$

and furthermore, from Burkholder-Davis-Gundy inequality, we have

$$E[|K_{t+\alpha}^{t,\zeta}|^2 | \mathcal{F}_t] \leq C E\left[\sup_{s \in [t, t+\alpha]} |X_s^{t,\zeta} - \zeta|^2 | \mathcal{F}_t \right] + C\alpha.$$

In view of (5.16), the proof is complete. \square

Remark 5.3. In view of (5.13) and (5.14), using Burkholder-Davis-Gundy inequality, we have

$$E\left[\sup_{s \in [t, t+\alpha]} |X_s^{t,\zeta} - \zeta|^8 | \mathcal{F}_t \right] \leq C\alpha^4. \quad (5.17)$$

Let us now define the random field:

$$u(t, x) = Y_s^{t,x}|_{s=t}, \quad (t, x) \in [0, T] \times \bar{D}, \quad (5.18)$$

where $Y^{t,x}$ is the solution of GBSDE (5.5) with $x \in \bar{D}$ at the place of $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \bar{D})$.

Proposition 5.2 yields that, for all $t \in [0, T]$, P-a.s.,

$$\begin{aligned}\text{(i)} \quad & |u(t, x) - u(t, y)| \leq C|x - y| + C|x - y|^{\frac{1}{2}}, \quad \text{for all } x, y \in \bar{D}; \\ \text{(ii)} \quad & |u(t, x)| \leq C(1 + |x|), \quad \text{for all } x \in \bar{D}.\end{aligned} \quad (5.19)$$

Theorem 5.1. Under the assumptions (H3.1) and (H3.2), for any $t \in [0, T]$ and $\zeta \in L^2(\Omega, \mathcal{F}_t, P; \bar{D})$, we have

$$u(t, \zeta) = Y_t^{t,\zeta}, \quad P\text{-a.s.} \quad (5.20)$$

The proof of Theorem 5.1 is similar to that of Theorem 3.1 in Peng [18] or Theorem A.2 in Buckdahn and Li [4]. Therefore it is omitted here.

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